МАТЕМАТИЧНЕ МОДЕЛЮВАННЯ ТА ОБЧИСЛЮВАЛЬНІ МЕТОДИ

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A THEOREM ON PLAYING THE STRICTLY CONVEX-CONCAVE CONTINUOUS ANTAGONISTIC GAME WITH THE SINGLE POSITIVE FIRST PLAYER COEFFICIENT AT ITS PURE STRATEGY IN THE KERNEL

Abstract: There has been proved the theorem on the pure strategies solution of the strictly convex-concave continuous antagonistic game, which kernel has the five nonzero parameters and the arbitrary constant. The solution is of the six different forms.
Анотація: Доведено теорему щодо розв'язку у чистих стратегіях строго опукло-вгнутої неперервної антагоністичної гри, чиє ядро має п'ять ненульових параметрів і довільну постійну. Розв'язок складається із шести різних форм.
Аннотация: Доказано теорему касательно решения в чистых стратегиях строго выпукло-вогнутой непрерывной антагонистической игры, чьё ядро имеет пять ненулевых параметров и произвольную постоянную. Решение состоит из шести различных форм.
Key words: pure strategies solution, convex-concave game, fair competition.

The problem decomposition and the paper investigation assignment

The strictly convex-concave continuous antagonistic game is an impartially appropriate and trustworthy mathematical model of some competitive activity socio-economic processes and events, attaching technical processes of machine-building aggregates run-in. The simplest type of the kernel of the strictly convex-concave continuous antagonistic game is the surface

$$K\left(x,\ y\right)=ax^2+bx+gxy+cy+hy^2+k\ , \tag{1}$$

defined on the unit square

$$S_K=X\times Y=\left[0;\ 1\right]\times\left[0;\ 1\right], \tag{2}$$

where $X=\left[0;\ 1\right]$ and $Y=\left[0;\ 1\right]$ are the sets of pure strategies of the first and second players respectively; $x\in X=\left[0;\ 1\right]$ and $y\in Y=\left[0;\ 1\right]$ are the pure strategies of the first and second players respectively; a, b, g, c and h are the real coefficients, and $k\in\mathbb{R}$ is a generalized constant. Such definition allows easily to propagate the consequent statements and conclusions for strictly convex-concave continuous antagonistic games with the kernel, defined generally on a Borelean subset $\boldsymbol{S}_K=X\times Y$ of the space $\mathbb{R}^2$ [1, 2], and modeling appropriate events [3], objects [1, 3], systems [2, 4, 5], processes [4, 5]. This paper investigation assignment is to prove the solution theorem for the strictly convex-concave continuous antagonistic game with the defined on the unit square (2) kernel (1) by some its nonzero coefficients, which are determined by their signs. In modeling processes of machine-building aggregates run-in time selection, where the first player is a stochastic unexpectedness, and the second is the run-in time (interval), it will ensure the reliable fixation of the run-in time.

Specifying the strictly convex-concave continuous antagonistic game and proving the theorem of its whole solution

It is known [6, 7], that if the continuous antagonistic game is strictly convex, then there is the condition

$$\frac{\partial^2 K\left(x,\ y\right)}{\partial y^2}>0\quad\forall\ x\in X\ \text{ and }\ \forall\ y\in Y\ . \tag{3}$$

Then for the kernel (1) there is the second partial derivative of y

$$\frac{\partial^2 K\left(x,\ y\right)}{\partial y^2}=\frac{\partial^2}{\partial y^2}\left(ax^2+bx+gxy+cy+hy^2+k\right)=\frac{\partial}{\partial y}\left(gx+c+2hy\right)=2h\ , \tag{4}$$

whence by the condition (3) here is $\dfrac{\partial^2 K\left(x,\ y\right)}{\partial y^2}=2h>0$ and the coefficient $h>0$. On the other part, if the continuous antagonistic game is strictly concave, then there is the condition

$$\frac{\partial^2 K\left(x,\ y\right)}{\partial x^2}<0\quad\forall\ x\in X\ \text{ and }\ \forall\ y\in Y\ . \tag{5}$$

The second partial derivative of x for the kernel (1) is

$$\frac{\partial^2 K\left(x,\ y\right)}{\partial x^2}=\frac{\partial^2}{\partial x^2}\left(ax^2+bx+gxy+cy+hy^2+k\right)=\frac{\partial}{\partial x}\left(2ax+b+gy\right)=2a\ , \tag{6}$$

whence by the condition (5) here is $\frac{\partial^2 K(x,\ y)}{\partial x^2} = 2a < 0$ and the coefficient $a < 0$. Consequently, there remain the coefficients b, g and c to be designated. Here may lay them down [8] as $b > 0$, $g < 0$, $c < 0$, that is the only positive first player coefficient is at its pure strategy x in the kernel (1).

For further acting, may the set $X_{\text{opt}} \subset X = [0;\ 1]$ be the optimal strategies set of the first player, and the set $Y_{\text{opt}} \subset Y = [0;\ 1]$ be the optimal strategies set of the second player in the specified game. In the pure strategies the optimal game solution [9, 10] will be stated as the set

$$\mathscr{R} = \left\{X_{\text{opt}},\ Y_{\text{opt}},\ V_{\text{opt}}\right\} \tag{7}$$

by the optimal game value V_{opt}. Then there is the theorem.

Theorem for the specified game solution

In the strictly convex-concave continuous antagonistic game with the kernel (1) by the parameters $a < 0$, $b > 0$, $g < 0$, $c < 0$, $h > 0$, and $k \in \mathbb{R}$, each of the players has the single optimal strategy, being the pure, where the number of the unique solutions equals to 6, and the pure strategy $y = 0$ of the second player must not be its optimal strategy.

Proof

As $a < 0$ then the parabola (1) being the function of the variable x has the global maximum point $x_{\max}$, which is the root of the equation [8]

$$\frac{\partial}{\partial x} K(x,\ y) = \frac{\partial}{\partial x}\left(ax^2 + bx + gxy + cy + hy^2 + k\right) = 2ax + b + gy = 0. \tag{8}$$

Then the point

$$x_{\max} = -\frac{b + gy}{2a} \tag{9}$$

and $x_{\max} \geqslant 0$ by $b + gy \geqslant 0$ or $y \leqslant -\frac{b}{g}$; $x_{\max} \leqslant 1$ by $b + gy \leqslant -2a$ or $y \geqslant -\frac{2a+b}{g}$. So, the global maximum point $x_{\max} \in X = [0;\ 1]$ if the point $y \in \left[-\frac{2a+b}{g};\ -\frac{b}{g}\right]$. Here the left end $-\frac{2a+b}{g} \geqslant 0$ if $2a + b \geqslant 0$ and the right end $-\frac{b}{g} \leqslant 1$ if $b + g \leqslant 0$.

As the point $x_{\max} \in X = [0;\ 1]$ then by $y \in \left[-\frac{2a+b}{g};\ -\frac{b}{g}\right]$ there is the kernel (1) maximum

$$\max_{x \in X} K(x,\ y) = \max_{x \in [0;\ 1]} K(x,\ y) = K(x_{\max},\ y) = K\left(-\frac{b+gy}{2a},\ y\right) =$$

$$= a\frac{(b+gy)^2}{4a^2} - b\frac{b+gy}{2a} - gy\frac{b+gy}{2a} + cy + hy^2 + k = -\frac{(b+gy)^2}{4a} + cy + hy^2 + k =$$

$$= -\frac{b^2}{4a} - \frac{bg}{2a}y + cy - \frac{g^2}{4a}y^2 + hy^2 + k = y^2\frac{4ah - g^2}{4a} + y\frac{2ac - bg}{2a} - \frac{b^2}{4a} + k, \tag{10}$$

where on the left end of the proper subsegment of $Y = [0;\ 1]$

$$K\left(x_{\max},\ -\frac{2a+b}{g}\right) = K\left(-\frac{b+gy}{2a},\ -\frac{2a+b}{g}\right) = K\left(1,\ -\frac{2a+b}{g}\right), \tag{11}$$

and on the right end

$$K\left(x_{\max},\ -\frac{b}{g}\right) = K\left(-\frac{b+gy}{2a},\ -\frac{b}{g}\right) = K\left(0,\ -\frac{b}{g}\right). \tag{12}$$

As there is the inequality $a + b + gy > 0$ by $y < -\frac{a+b}{g}$ and $-\frac{a+b}{g} \in \left(-\frac{2a+b}{g};\ -\frac{b}{g}\right)$ then the maximum of the surface (1) on the unit segment $X = [0;\ 1]$ of the variable x is

$$\max_{x\in[0;\,1]} K(x,\,y)=\begin{cases}\max\{K(0,\,y),\,K(1,\,y)\}=K(1,\,y)=a+b+gy+cy+hy^2+k,\ y\in\left[0;\,-\dfrac{2a+b}{g}\right];\\ K(x_{\max},\,y)=K\left(-\dfrac{b+gy}{2a},\,y\right)=y^2\dfrac{4ah-g^2}{4a}+y\dfrac{2ac-bg}{2a}-\dfrac{b^2}{4a}+k,\ y\in\left[-\dfrac{2a+b}{g};\,-\dfrac{b}{g}\right];\\ \max\{K(0,\,y),\,K(1,\,y)\}=K(0,\,y)=cy+hy^2+k,\ y\in\left[-\dfrac{b}{g};\,1\right].\end{cases} \quad (13)$$

For minimizing the function (13) on the unit segment $Y=[0;\,1]$ it is necessary to determine the minima of the three parabolas $K(1,\,y)$, $K(x_{\max},\,y)$ and $K(0,\,y)$ on the specified in (13) three subsegments of Y. The global minimum $y_{\min}^{\langle 1\rangle}$ of the parabola $K(1,\,y)$ is the root of the equation

$$\frac{d}{dy}K(1,\,y)=\frac{d}{dy}\left(a+b+gy+cy+hy^2+k\right)=g+c+2hy=0\,, \quad (14)$$

whence the point

$$y_{\min}^{\langle 1\rangle}=-\frac{g+c}{2h}\,. \quad (15)$$

As $g+c<0$ then the point $y_{\min}^{\langle 1\rangle}>0$ and $y_{\min}^{\langle 1\rangle}<-\dfrac{2a+b}{g}$ if there is the statement

$$y_{\min}^{\langle 1\rangle}-\left(-\frac{2a+b}{g}\right)=-\frac{g+c}{2h}+\frac{2a+b}{g}=\frac{2h(2a+b)-g(g+c)}{2hg}<0\,, \quad (16)$$

whence $y_{\min}^{\langle 1\rangle}\in\left(0;\,-\dfrac{2a+b}{g}\right)$ by $2h(2a+b)-g(g+c)>0$. Mark, that the value of the parabola $K(1,\,y)$ in the point $y_{\min}^{\langle 1\rangle}$ is

$$K\left(1,\,y_{\min}^{\langle 1\rangle}\right)=K\left(1,\,-\frac{g+c}{2h}\right)=a+b+g\left(-\frac{g+c}{2h}\right)+c\left(-\frac{g+c}{2h}\right)+h\left(-\frac{g+c}{2h}\right)^2+k=a+b+$$
$$+(g+c)\left(-\frac{g+c}{2h}\right)+h\left(-\frac{g+c}{2h}\right)^2+k=a+b-\frac{(g+c)^2}{2h}+\frac{(g+c)^2}{4h}+k=a+b-\frac{(g+c)^2}{4h}+k\,. \quad (17)$$

Turning to the parabola $K(x_{\max},\,y)$, it is seen that the coefficient $\dfrac{4ah-g^2}{4a}>0$ points at that the parabola $K(x_{\max},\,y)$ has the global minimum $y_{\min}$, that is the root of the equation

$$\frac{d}{dy}K(x_{\max},\,y)=\frac{d}{dy}\left(y^2\frac{4ah-g^2}{4a}+y\frac{2ac-bg}{2a}-\frac{b^2}{4a}+k\right)=y\frac{4ah-g^2}{2a}+\frac{2ac-bg}{2a}=0\,, \quad (18)$$

whence the point

$$y_{\min}=\frac{bg-2ac}{4ah-g^2} \quad (19)$$

and the function (10) in this point is

$$K(x_{\max},\,y_{\min})=K\left(-\frac{b+gy}{2a},\,\frac{bg-2ac}{4ah-g^2}\right)=\left(\frac{bg-2ac}{4ah-g^2}\right)^2\frac{4ah-g^2}{4a}+\frac{bg-2ac}{4ah-g^2}\frac{2ac-bg}{2a}-\frac{b^2}{4a}+k=$$
$$=\frac{(bg-2ac)^2}{4a\left(4ah-g^2\right)}-\frac{(bg-2ac)^2}{2a\left(4ah-g^2\right)}-\frac{b^2}{4a}+k=-\frac{(bg-2ac)^2}{4a\left(4ah-g^2\right)}-\frac{b^2}{4a}+k=$$
$$=\frac{b^2g^2-4acbg+4a^2c^2-b^2g^2+4ab^2h}{4a\left(g^2-4ah\right)}+k=\frac{-cbg+ac^2+b^2h}{g^2-4ah}+k=\frac{c(ac-bg)+b^2h}{g^2-4ah}+k\,. \quad (20)$$

Obviously, that by the initial conditions there are the inequalities $bg-2ac<0$ and $4ah-g^2<0$, whence the point $y_{\min}=\dfrac{bg-2ac}{4ah-g^2}>0$. The difference between the point (19) and the left end of the subsegment $\left[-\dfrac{2a+b}{g};\,-\dfrac{b}{g}\right]$ is

$$y_{\min} - \left(-\frac{2a+b}{g}\right) = \frac{bg-2ac}{4ah-g^2} + \frac{2a+b}{g} = \frac{bg^2 - 2acg + \left(4ah-g^2\right)\left(2a+b\right)}{g\left(4ah-g^2\right)} =$$

$$= \frac{2a\left(2bh-cg\right) + 2a\left(4ah-g^2\right)}{g\left(4ah-g^2\right)} = \frac{2a}{g\left(4ah-g^2\right)}\left[2h\left(2a+b\right) - g\left(g+c\right)\right]. \tag{21}$$

So, $y_{\min} > -\frac{2a+b}{g}$ by $2h\left(2a+b\right) - g\left(g+c\right) < 0$. And as the difference

$$y_{\min} - \left(-\frac{b}{g}\right) = \frac{bg-2ac}{4ah-g^2} + \frac{b}{g} = \frac{bg^2 - 2acg + 4ahb - bg^2}{g\left(4ah-g^2\right)} = \frac{2a}{g\left(4ah-g^2\right)}\left(2hb-cg\right) \tag{22}$$

then $y_{\min} < -\frac{b}{g}$ by $2hb - cg > 0$.

The global minimum $y_{\min}^{\langle 0\rangle}$ of the parabola $K\left(0,\, y\right)$ is the root of the equation

$$\frac{d}{dy}K\left(0,\, y\right) = \frac{d}{dy}\left(cy + hy^2 + k\right) = c + 2hy = 0, \tag{23}$$

whence the point

$$y_{\min}^{\langle 0\rangle} = -\frac{c}{2h}. \tag{24}$$

Certainly that $-\frac{c}{2h} > 0$ and as the difference

$$y_{\min}^{\langle 0\rangle} - \left(-\frac{b}{g}\right) = -\frac{c}{2h} + \frac{b}{g} = \frac{2hb-cg}{2hg} \tag{25}$$

then $y_{\min}^{\langle 0\rangle} > -\frac{b}{g}$ by $2hb - cg < 0$; and $y_{\min}^{\langle 0\rangle} \leqslant 1$ by $c + 2h \geqslant 0$.

For further proving there should be considered all the cases with the inequalities, defining every saddle point.

Case 1. $b + g \leqslant 0$, $2a + b \geqslant 0$, $2h\left(2a+b\right) - g\left(g+c\right) > 0$

Here is the point $y_{\min}^{\langle 1\rangle} \in \left(0;\, -\frac{2a+b}{g}\right)$. The difference between the points $y_{\min}^{\langle 1\rangle}$ and $y_{\min}^{\langle 0\rangle}$ is

$$y_{\min}^{\langle 1\rangle} - y_{\min}^{\langle 0\rangle} = -\frac{g+c}{2h} - \left(-\frac{c}{2h}\right) = -\frac{g}{2h} > 0. \tag{26}$$

As the point $y_{\min}^{\langle 1\rangle} < -\frac{2a+b}{g}$ then $y_{\min}^{\langle 0\rangle} < -\frac{2a+b}{g}$ and $y_{\min}^{\langle 0\rangle} < -\frac{b}{g}$. Also here the point $y_{\min} < -\frac{2a+b}{g}$. Applying the double parabolic inequalities

$$K\left(0,\, y_{\min}^{\langle 0\rangle}\right) < K\left(0,\, -\frac{b}{g}\right) < K\left(0,\, 1\right), \tag{27}$$

$$K\left(x_{\max},\, y_{\min}\right) < K\left(x_{\max},\, -\frac{2a+b}{g}\right) < K\left(x_{\max},\, -\frac{b}{g}\right) = K\left(0,\, -\frac{b}{g}\right), \tag{28}$$

$$\left\{K\left(1,\, 0\right) > K\left(1,\, y_{\min}^{\langle 1\rangle}\right),\; K\left(1,\, y_{\min}^{\langle 1\rangle}\right) < K\left(1,\, -\frac{2a+b}{g}\right) = K\left(x_{\max},\, -\frac{2a+b}{g}\right)\right\}, \tag{29}$$

the minimum of the function (13) on the segment Y

$$\min_{y\in[0;\,1]}\max_{x\in[0;\,1]} K\left(x,\, y\right) = \min\left\{\min_{y\in\left[0;\, -\frac{2a+b}{g}\right]} K\left(1,\, y\right),\; \min_{y\in\left[-\frac{2a+b}{g};\, -\frac{b}{g}\right]} K\left(x_{\max},\, y\right),\; \min_{y\in\left[-\frac{b}{g};\, 1\right]} K\left(0,\, y\right)\right\} =$$

$$= \min\left\{K\left(1,\, y_{\min}^{\langle 1\rangle}\right),\; \min\left\{K\left(x_{\max},\, -\frac{2a+b}{g}\right),\; K\left(x_{\max},\, -\frac{b}{g}\right)\right\},\; \min\left\{K\left(0,\, -\frac{b}{g}\right),\; K\left(0,\, 1\right)\right\}\right\} =$$

$$= \min\left\{K\left(1,\, y_{\min}^{\langle 1\rangle}\right),\, K\left(x_{\max},\, -\frac{2a+b}{g}\right),\, K\left(0,\, -\frac{b}{g}\right)\right\} =$$

$$= \min\left\{K\left(1,\, y_{\min}^{\langle 1\rangle}\right),\, K\left(x_{\max},\, -\frac{2a+b}{g}\right),\, K\left(x_{\max},\, -\frac{b}{g}\right)\right\} =$$

$$= \min\left\{K\left(1,\, y_{\min}^{\langle 1\rangle}\right),\, K\left(x_{\max},\, -\frac{2a+b}{g}\right)\right\} = K\left(1,\, y_{\min}^{\langle 1\rangle}\right) = a+b-\frac{(g+c)^2}{4h}+k = V_{\text{opt}} \tag{30}$$

is reached on the optimal strategies set of the second player

$$Y_{\text{opt}} = \left\{y_{\min}^{\langle 1\rangle}\right\} = \left\{-\frac{g+c}{2h}\right\} = \left\{y_{\text{opt}}\right\}. \tag{31}$$

The optimal strategies set X_{opt} of the first player is determined by the roots x_1 and x_2 of the quadratic equation [6, 7]

$$V_{\text{opt}} = K\left(x,\, y_{\text{opt}}\right). \tag{32}$$

In the case being investigated the corresponding equation (32) is

$$V_{\text{opt}} = K\left(1,\, -\frac{g+c}{2h}\right) = a+b-\frac{(g+c)^2}{4h}+k = ax^2+bx+gx\left(-\frac{g+c}{2h}\right)+c\left(-\frac{g+c}{2h}\right)+h\left(-\frac{g+c}{2h}\right)^2 +$$

$$+k = ax^2 + x\left[\frac{2bh-g(g+c)}{2h}\right] - c\frac{g+c}{2h} + \frac{(g+c)^2}{4h} + k = K\left(x,\, -\frac{g+c}{2h}\right) = K\left(x,\, y_{\text{opt}}\right); \tag{33}$$

$$ax^2 + x\left[\frac{2bh-g(g+c)}{2h}\right] - c\frac{g+c}{2h} + \frac{(g+c)^2}{4h} - a - b + \frac{(g+c)^2}{4h} =$$

$$= ax^2 + x\left[\frac{2bh-g(g+c)}{2h}\right] + \frac{g(g+c)-2h(a+b)}{2h} =$$

$$= a\left(x^2 + x\left[\frac{2bh-g(g+c)}{2ah}\right] + \frac{g(g+c)-2h(a+b)}{2ah}\right) = a(x-1)\left(x - \frac{g(g+c)-2h(a+b)}{2ah}\right) = 0. \tag{34}$$

It is seen from the statement (34) that the roots of the equation (33) are $x_1 = 1$ and $x_2 = \frac{g(g+c)-2h(a+b)}{2ah}$. But as $2h(a+b)-g(g+c) > -2ha > 0$ then $\frac{g(g+c)-2h(a+b)}{2ah} > 1$ and the optimal strategies set of the first player is

$$X_{\text{opt}} = \{x_1\} = \{1\}. \tag{35}$$

Accordingly the investigated case game solution is

$$\mathscr{R} = \left\{\{1\},\, \left\{-\frac{g+c}{2h}\right\},\, a+b-\frac{(g+c)^2}{4h}+k\right\}. \tag{36}$$

The next case must be at $2h(2a+b)-g(g+c) \leqslant 0$, where $y_{\min}^{\langle 1\rangle} \geqslant -\frac{2a+b}{g}$ and $y_{\min} \geqslant -\frac{2a+b}{g}$. Setting on the inequality $2hb-cg > 0$ there due to (22) is the point $y_{\min} < -\frac{b}{g}$. And due to (25) there is the point $y_{\min}^{\langle 0\rangle} < -\frac{b}{g}$.

Case 2. $b+g \leqslant 0$, $2a+b \geqslant 0$, $2h(2a+b)-g(g+c) \leqslant 0$, $2hb-cg > 0$

Having the point $y_{\min} \in \left[-\frac{2a+b}{g};\, -\frac{b}{g}\right)$, the corresponding double parabolic inequalities (27) and

$$\left\{K\left(x_{\max},\, -\frac{2a+b}{g}\right) \geqslant K\left(x_{\max},\, y_{\min}\right),\, K\left(x_{\max},\, y_{\min}\right) < K\left(x_{\max},\, -\frac{b}{g}\right) = K\left(0,\, -\frac{b}{g}\right)\right\}, \tag{37}$$

$$K(1,\, 0) > K\left(1,\, -\frac{2a+b}{g}\right) = K\left(x_{\max},\, -\frac{2a+b}{g}\right) \geqslant K\left(1,\, y_{\min}^{\langle 1\rangle}\right), \tag{38}$$

the minimum of the function (13) on the segment Y

$$\min_{y\in[0;\,1]}\max_{x\in[0;\,1]}K(x,\,y)=\min\left\{\min_{y\in\left[0;\,-\frac{2a+b}{g}\right]}K(1,\,y),\ \min_{y\in\left[-\frac{2a+b}{g};\,-\frac{b}{g}\right]}K(x_{\max},\,y),\ \min_{y\in\left[-\frac{b}{g};\,1\right]}K(0,\,y)\right\}=$$

$$=\min\left\{\min\left\{K(1,\,0),\,K\left(1,\,-\frac{2a+b}{g}\right)\right\},\,K(x_{\max},\,y_{\min}),\,\min\left\{K\left(0,\,-\frac{b}{g}\right),\,K(0,\,1)\right\}\right\}=$$

$$=\min\left\{K\left(1,\,-\frac{2a+b}{g}\right),\,K(x_{\max},\,y_{\min}),\,K\left(0,\,-\frac{b}{g}\right)\right\}=$$

$$=K(x_{\max},\,y_{\min})=K\left(-\frac{b+gy}{2a},\,\frac{bg-2ac}{4ah-g^2}\right)=\frac{c(ac-bg)+b^2h}{g^2-4ah}+k=V_{\text{opt}} \qquad (39)$$

is reached on the optimal strategies set of the second player

$$Y_{\text{opt}}=\{y_{\min}\}=\left\{\frac{bg-2ac}{4ah-g^2}\right\}=\{y_{\text{opt}}\}. \qquad (40)$$

The corresponding equation (32) is

$$V_{\text{opt}}=K\left(-\frac{b+gy}{2a},\,\frac{bg-2ac}{4ah-g^2}\right)=\frac{c(ac-bg)+b^2h}{g^2-4ah}+k=$$

$$=ax^2+bx+gx\frac{bg-2ac}{4ah-g^2}+c\frac{bg-2ac}{4ah-g^2}+h\left(\frac{bg-2ac}{4ah-g^2}\right)^2+k=K\left(x,\,\frac{bg-2ac}{4ah-g^2}\right)=K(x,\,y_{\text{opt}}); \qquad (41)$$

$$ax^2+bx+gx\frac{bg-2ac}{4ah-g^2}+c\frac{bg-2ac}{4ah-g^2}+h\left(\frac{bg-2ac}{4ah-g^2}\right)^2-\frac{c(ac-bg)+b^2h}{g^2-4ah}=$$

$$=ax^2+x\frac{4abh-bg^2+bg^2-2acg}{4ah-g^2}+\frac{(bcg-2ac^2)(4ah-g^2)+h(bg-2ac)^2}{(4ah-g^2)^2}+\frac{c(ac-bg)+b^2h}{4ah-g^2}=$$

$$=ax^2+x\frac{2a(2bh-cg)}{4ah-g^2}+\frac{4ahbcg-8a^2c^2h-bcg^3+2ac^2g^2+hb^2g^2-4ahbcg+4a^2c^2h}{(4ah-g^2)^2}+$$

$$+\frac{\left[c(ac-bg)+b^2h\right](4ah-g^2)}{(4ah-g^2)^2}=ax^2+x\frac{2a(2bh-cg)}{4ah-g^2}+\frac{-bcg^3+2ac^2g^2+hb^2g^2-4a^2c^2h}{(4ah-g^2)^2}+$$

$$+\frac{4a^2c^2h-4abcgh+4ab^2h^2-ac^2g^2+bcg^3-b^2hg^2}{(4ah-g^2)^2}=$$

$$=ax^2+x\frac{2a(2bh-cg)}{4ah-g^2}+\frac{ac^2g^2-4abcgh+4ab^2h^2}{(4ah-g^2)^2}=ax^2+x\frac{2a(2bh-cg)}{4ah-g^2}+$$

$$+a\frac{(cg-2bh)^2}{(4ah-g^2)^2}=a\left[x^2+2x\frac{2bh-cg}{4ah-g^2}+\left(\frac{cg-2bh}{4ah-g^2}\right)^2\right]=a\left(x+\frac{2bh-cg}{4ah-g^2}\right)^2=0. \qquad (42)$$

Essentially, that with the statement (42) the single root of the equation (41) is $x_1=x_2=\frac{cg-2bh}{4ah-g^2}$. As $2hb-cg>0$ then $\frac{cg-2bh}{4ah-g^2}>0$, and inasmuch $2h(2a+b)-g(g+c)\leqslant 0$ then $4ah-g^2\leqslant cg-2bh$, whence the value $\frac{cg-2bh}{4ah-g^2}\in(0;\,1]$. So, the optimal strategies set of the first player here is

$$X_{\text{opt}}=\{x_1\}=\{x_2\}=\left\{\frac{cg-2bh}{4ah-g^2}\right\}. \qquad (43)$$

Accordingly the investigated case game solution is

$$\mathscr{R}=\left\{\left\{\frac{cg-2bh}{4ah-g^2}\right\},\left\{\frac{bg-2ac}{4ah-g^2}\right\},\frac{c\left(ac-bg\right)+b^2h}{g^2-4ah}+k\right\}. \tag{44}$$

The subsequent case is to be investigated by $2hb-cg\leqslant 0$, when still $y_{\min}^{\langle 1\rangle}\geqslant-\frac{2a+b}{g}$ and the point $y_{\min}\geqslant-\frac{b}{g}$. Besides, due to (25) there is the point $y_{\min}^{\langle 0\rangle}\geqslant-\frac{b}{g}$, where it should be attached the inequality $c+2h\geqslant 0$, jointly giving the point $y_{\min}^{\langle 0\rangle}\in\left[-\frac{b}{g};\,1\right]$.

Case 3. $b+g\leqslant 0$, $2a+b\geqslant 0$, $2h\left(2a+b\right)-g\left(g+c\right)\leqslant 0$, $2hb-cg\leqslant 0$, $c+2h\geqslant 0$

Here are the double parabolic inequalities

$$\left\{K\left(0,\,-\frac{b}{g}\right)\geqslant K\left(0,\,y_{\min}^{\langle 0\rangle}\right),\,K\left(0,\,y_{\min}^{\langle 0\rangle}\right)\leqslant K\left(0,\,1\right)\right\}, \tag{45}$$

$$K\left(x_{\max},\,-\frac{2a+b}{g}\right)>K\left(x_{\max},\,-\frac{b}{g}\right)=K\left(0,\,-\frac{b}{g}\right)\geqslant K\left(x_{\max},\,y_{\min}\right), \tag{46}$$

and (38). Then the minimum of the function (13) on the segment Y

$$\min_{y\in[0;\,1]}\max_{x\in[0;\,1]}K\left(x,\,y\right)=\min\left\{\min_{y\in\left[0;\,-\frac{2a+b}{g}\right]}K\left(1,\,y\right),\ \min_{y\in\left[-\frac{2a+b}{g};\,-\frac{b}{g}\right]}K\left(x_{\max},\,y\right),\ \min_{y\in\left[-\frac{b}{g};\,1\right]}K\left(0,\,y\right)\right\}=$$

$$=\min\left\{\min\left\{K\left(1,\,0\right),\,K\left(1,\,-\frac{2a+b}{g}\right)\right\},\,\min\left\{K\left(x_{\max},\,-\frac{2a+b}{g}\right),\,K\left(x_{\max},\,-\frac{b}{g}\right)\right\},\,K\left(0,\,y_{\min}^{\langle 0\rangle}\right)\right\}=$$

$$=\min\left\{K\left(1,\,-\frac{2a+b}{g}\right),\,K\left(x_{\max},\,-\frac{b}{g}\right),\,K\left(0,\,y_{\min}^{\langle 0\rangle}\right)\right\}=$$

$$=\min\left\{K\left(x_{\max},\,-\frac{2a+b}{g}\right),\,K\left(x_{\max},\,-\frac{b}{g}\right),\,K\left(0,\,y_{\min}^{\langle 0\rangle}\right)\right\}=$$

$$=\min\left\{K\left(x_{\max},\,-\frac{b}{g}\right),\,K\left(0,\,y_{\min}^{\langle 0\rangle}\right)\right\}=\min\left\{K\left(0,\,-\frac{b}{g}\right),\,K\left(0,\,y_{\min}^{\langle 0\rangle}\right)\right\}=K\left(0,\,y_{\min}^{\langle 0\rangle}\right)=$$

$$=K\left(0,\,-\frac{c}{2h}\right)=c\left(-\frac{c}{2h}\right)+h\left(-\frac{c}{2h}\right)^2+k=k-\frac{c^2}{4h}=V_{\text{opt}} \tag{47}$$

is reached on the optimal strategies set of the second player

$$Y_{\text{opt}}=\left\{y_{\min}^{\langle 0\rangle}\right\}=\left\{-\frac{c}{2h}\right\}=\left\{y_{\text{opt}}\right\}. \tag{48}$$

The corresponding equation (32) is

$$V_{\text{opt}}=K\left(0,\,-\frac{c}{2h}\right)=k-\frac{c^2}{4h}=ax^2+bx+gx\left(-\frac{c}{2h}\right)+c\left(-\frac{c}{2h}\right)+h\left(-\frac{c}{2h}\right)^2+k=$$

$$=ax^2+bx+gx\left(-\frac{c}{2h}\right)+k-\frac{c^2}{4h}=x\left(ax+b-\frac{cg}{2h}\right)+k-\frac{c^2}{4h}=$$

$$=x\left(ax+\frac{2hb-cg}{2h}\right)+k-\frac{c^2}{4h}=K\left(x,\,-\frac{c}{2h}\right)=K\left(x,\,y_{\text{opt}}\right). \tag{49}$$

From the equation (49) get the equation

$$x\left(ax+\frac{2hb-cg}{2h}\right)=x\left(x+\frac{2hb-cg}{2ah}\right)=0, \tag{50}$$

whence the roots of the equation (49) are $x_1=0$ and $x_2=\frac{cg-2hb}{2ah}$. By the initial condition $2hb-cg\leqslant 0$ it means that $x_2=\frac{cg-2hb}{2ah}\leqslant 0$. So, the optimal strategies set of the first player is

$$X_{\text{opt}}=\left\{x_1\right\}=\left\{0\right\}. \tag{51}$$

Accordingly this local investigated case game solution is the set

$$\mathscr{R} = \left\{ \{0\}, \left\{ -\frac{c}{2h} \right\}, k - \frac{c^2}{4h} \right\}. \tag{52}$$

Case 4. $b+g \leqslant 0$, $2a+b \geqslant 0$, $2h(2a+b) - g(g+c) \leqslant 0$, $2hb - cg \leqslant 0$, $c + 2h < 0$

There are the points $y_{\min}^{\langle 1 \rangle} \geqslant -\frac{2a+b}{g}$, $y_{\min} \geqslant -\frac{b}{g}$, $y_{\min}^{\langle 0 \rangle} > 1$, satisfying the double parabolic inequalities

$$K\left(0, -\frac{b}{g}\right) > K(0, 1) > K\left(0, y_{\min}^{\langle 0 \rangle}\right), \tag{53}$$

(46) and (38). Then the minimum of the function (13) on the segment Y

$$\min_{y \in [0;1]} \max_{x \in [0;1]} K(x, y) = \min\left\{ \min_{y \in \left[0; -\frac{2a+b}{g}\right]} K(1, y), \min_{y \in \left[-\frac{2a+b}{g}; -\frac{b}{g}\right]} K(x_{\max}, y), \min_{y \in \left[-\frac{b}{g}; 1\right]} K(0, y) \right\} =$$

$$= \min\left\{ \min\left\{ K(1, 0), K\left(1, -\frac{2a+b}{g}\right) \right\}, \min\left\{ K\left(x_{\max}, -\frac{2a+b}{g}\right), K\left(x_{\max}, -\frac{b}{g}\right) \right\}, \min\left\{ K\left(0, -\frac{b}{g}\right), K(0, 1) \right\} \right\} =$$

$$= \min\left\{ K\left(1, -\frac{2a+b}{g}\right), K\left(x_{\max}, -\frac{b}{g}\right), K(0, 1) \right\} =$$

$$= \min\left\{ K\left(x_{\max}, -\frac{2a+b}{g}\right), K\left(x_{\max}, -\frac{b}{g}\right), K(0, 1) \right\} =$$

$$= \min\left\{ K\left(x_{\max}, -\frac{b}{g}\right), K(0, 1) \right\} = \min\left\{ K\left(0, -\frac{b}{g}\right), K(0, 1) \right\} = K(0, 1) = c + h + k = V_{\text{opt}} \tag{54}$$

is reached on the optimal strategies set of the second player

$$Y_{\text{opt}} = \{1\} = \{y_{\text{opt}}\}. \tag{55}$$

The roots of the corresponding equation (32)

$$V_{\text{opt}} = K(0, 1) = c + h + k = ax^2 + bx + gx + c + h + k =$$

$$= ax\left(x + \frac{b+g}{a}\right) + c + h + k = K(x, 1) = K(x, y_{\text{opt}}) \tag{56}$$

are $x_1 = 0$ and $x_2 = -\frac{b+g}{a}$. But $b+g \leqslant 0$ means $x_2 = -\frac{b+g}{a} \leqslant 0$, so the optimal strategies set of the first player is (51). Accordingly this local investigated case game solution is the set

$$\mathscr{R} = \{\{0\}, \{1\}, c + h + k\}. \tag{57}$$

At this the maximum (13) has been investigated completely. For subsequent investigations there should be set up the condition $2a + b < 0$. Right away the value $-\frac{2a+b}{g} < 0$ and the maximum of the surface (1) on the unit segment $X = [0; 1]$ of the variable x is

$$\max_{x \in [0;1]} K(x, y) = \begin{cases} K(x_{\max}, y) = K\left(-\frac{b+gy}{2a}, y\right) = y^2 \frac{4ah - g^2}{4a} + y\frac{2ac - bg}{2a} - \frac{b^2}{4a} + k, \ y \in \left[0; -\frac{b}{g}\right]; \\ \max\{K(0, y), K(1, y)\} = K(0, y) = cy + hy^2 + k, \ y \in \left[-\frac{b}{g}; 1\right]. \end{cases} \tag{58}$$

If to set there up the condition $2hb - cg > 0$, then the statement (22) will give $y_{\min} < -\frac{b}{g}$, and the statement (25) will give $y_{\min}^{\langle 0 \rangle} < -\frac{b}{g}$.

Case 5. $b + g \leqslant 0$, $2a + b < 0$, $2hb - cg > 0$

Here is the point $y_{\min} \in \left(0; -\frac{b}{g}\right)$, giving the corresponding double parabolic inequality

$$\left\{K\left(x_{\max}, 0\right) > K\left(x_{\max}, y_{\min}\right), K\left(x_{\max}, y_{\min}\right) < K\left(x_{\max}, -\frac{b}{g}\right) = K\left(0, -\frac{b}{g}\right)\right\}. \quad (59)$$

With the double parabolic inequality (27) the minimum of the function (58) on the segment Y

$$\min_{y\in[0;\,1]}\max_{x\in[0;\,1]} K\left(x, y\right) = \min\left\{\min_{y\in\left[0;\,-\frac{b}{g}\right]} K\left(x_{\max}, y\right), \min_{y\in\left[-\frac{b}{g};\,1\right]} K\left(0, y\right)\right\} =$$

$$= \min\left\{K\left(x_{\max}, y_{\min}\right), \min\left\{K\left(0, -\frac{b}{g}\right), K\left(0, 1\right)\right\}\right\} =$$

$$= \min\left\{K\left(x_{\max}, y_{\min}\right), K\left(0, -\frac{b}{g}\right)\right\} = \min\left\{K\left(x_{\max}, y_{\min}\right), K\left(x_{\max}, -\frac{b}{g}\right)\right\} =$$

$$= K\left(x_{\max}, y_{\min}\right) = K\left(-\frac{b+gy}{2a}, \frac{bg-2ac}{4ah-g^2}\right) = \frac{c\left(ac-bg\right)+b^2h}{g^2-4ah} + k = V_{\text{opt}} \quad (60)$$

is reached on the optimal strategies set (40) of the second player. The single root $x_1 = x_2 = \frac{cg-2bh}{4ah-g^2}$ of the corresponding equation (41) is determined by the equation (42). As $2hb - cg > 0$ then again $\frac{cg-2bh}{4ah-g^2} > 0$, and inasmuch $2h\left(2a+b\right) - g\left(g+c\right) < 0$ then $4ah - g^2 < cg - 2bh$, whence the value $\frac{cg-2bh}{4ah-g^2} \in \left(0; 1\right)$. So, the set (43) is the optimal strategies set of the first player, and the investigated case game solution is the set (44).

Further setting up the inequality $2hb - cg \leqslant 0$ drives to that the point $y_{\min} \geqslant -\frac{b}{g}$, but $y_{\min}^{\langle 0\rangle} \in \left[-\frac{b}{g}; 1\right]$ by the inequality $c + 2h \geqslant 0$.

Case 6. $b + g \leqslant 0$, $2a + b < 0$, $2hb - cg \leqslant 0$, $c + 2h \geqslant 0$

Here are the point $y_{\min}^{\langle 0\rangle} \in \left[-\frac{b}{g}; 1\right]$ and the point $y_{\min} \geqslant -\frac{b}{g}$, giving the double parabolic inequalities (45) and

$$K\left(x_{\max}, 0\right) > K\left(x_{\max}, -\frac{b}{g}\right) = K\left(0, -\frac{b}{g}\right) \geqslant K\left(x_{\max}, y_{\min}\right). \quad (61)$$

Then the minimum of the function (58) on the segment Y

$$\min_{y\in[0;\,1]}\max_{x\in[0;\,1]} K\left(x, y\right) = \min\left\{\min_{y\in\left[0;\,-\frac{b}{g}\right]} K\left(x_{\max}, y\right), \min_{y\in\left[-\frac{b}{g};\,1\right]} K\left(0, y\right)\right\} =$$

$$= \min\left\{\min\left\{K\left(x_{\max}, 0\right), K\left(x_{\max}, -\frac{b}{g}\right)\right\}, K\left(0, y_{\min}^{\langle 0\rangle}\right)\right\} =$$

$$= \min\left\{K\left(x_{\max}, -\frac{b}{g}\right), K\left(0, y_{\min}^{\langle 0\rangle}\right)\right\} = \min\left\{K\left(0, -\frac{b}{g}\right), K\left(0, y_{\min}^{\langle 0\rangle}\right)\right\} =$$

$$= K\left(0, y_{\min}^{\langle 0\rangle}\right) = K\left(0, -\frac{c}{2h}\right) = k - \frac{c^2}{4h} = V_{\text{opt}} \quad (62)$$

is reached on the optimal strategies set (48) of the second player. The roots of the corresponding equation (49) are $x_1 = 0$ and $x_2 = \frac{cg-2hb}{2ah} \leqslant 0$, that is the optimal strategies set of the first player is the set (51). Accordingly this local investigated case game solution is the set (52).

Case 7. $b + g \leqslant 0$, $2a + b < 0$, $2hb - cg \leqslant 0$, $c + 2h < 0$

As the point $y_{\min} \geqslant -\frac{b}{g}$ and the point $y_{\min}^{\langle 0 \rangle} > 1$ then respectively with the double parabolic inequalities (61) and (53) there the segment Y function (58) minimum

$$\min_{y \in [0;\, 1]} \max_{x \in [0;\, 1]} K(x,\, y) = \min \left\{ \min_{y \in \left[0;\, -\frac{b}{g}\right]} K(x_{\max},\, y),\ \min_{y \in \left[-\frac{b}{g};\, 1\right]} K(0,\, y) \right\} =$$

$$= \min \left\{ \min \left\{ K(x_{\max},\, 0),\ K\left(x_{\max},\, -\frac{b}{g}\right) \right\},\ \min \left\{ K\left(0,\, -\frac{b}{g}\right),\ K(0,\, 1) \right\} \right\} =$$

$$= \min \left\{ K\left(x_{\max},\, -\frac{b}{g}\right),\ K(0,\, 1) \right\} = \min \left\{ K\left(0,\, -\frac{b}{g}\right),\ K(0,\, 1) \right\} = K(0,\, 1) = c + h + k = V_{\text{opt}} \tag{63}$$

is reached on the set (55). The roots of the corresponding equation (56) are $x_1 = 0$ and $x_2 = -\frac{b+g}{a} \leqslant 0$, that is the optimal strategies set of the first player is (51). Accordingly this local investigated case game solution is the set (57).

At this the maximum (58), or, to be more general, the maximum (13) by sum $b + g \leqslant 0$, has been investigated completely. For subsequent investigations there should be set up the condition $b + g > 0$. As $-\frac{b}{g} > 1$ then $x_{\max} \geqslant 0$ by $y \leqslant 1$, and if $2a + b \geqslant 0$ then $x_{\max} \leqslant 1$ by $y \geqslant -\frac{2a+b}{g}$. At this the value $-\frac{2a+b}{g} \in [0;\, 1]$ by $2a + b + g \leqslant 0$. The inequality $a + b + gy > 0$ by $y < -\frac{a+b}{g}$ means that the maximum of the surface (1) on the unit segment $X = [0;\, 1]$ of the variable x is

$$\max_{x \in [0;\, 1]} K(x,\, y) = \begin{cases} \max\{K(0,\, y),\, K(1,\, y)\} = K(1,\, y) = a + b + gy + cy + hy^2 + k,\ y \in \left[0;\, -\frac{2a+b}{g}\right]; \\ K(x_{\max},\, y) = K\left(-\frac{b+gy}{2a},\, y\right) = y^2 \frac{4ah - g^2}{4a} + y \frac{2ac - bg}{2a} - \frac{b^2}{4a} + k,\ y \in \left[-\frac{2a+b}{g};\, 1\right]. \end{cases} \tag{64}$$

Due to the inequality (16) there by the condition $2h(2a+b) - g(g+c) > 0$ will be the point $y_{\min}^{\langle 1 \rangle} \in \left(0;\, -\frac{2a+b}{g}\right)$, and due to the statement (21) there by the same condition will be the point $y_{\min} < -\frac{2a+b}{g}$.

Case 8. $b + g > 0$, $2a + b \geqslant 0$, $2a + b + g \leqslant 0$, $2h(2a+b) - g(g+c) > 0$

Having here the double parabolic inequalities (29) and

$$K(x_{\max},\, y_{\min}) < K\left(x_{\max},\, -\frac{2a+b}{g}\right) < K(x_{\max},\, 1), \tag{65}$$

the minimum of the function (64) on the segment Y

$$\min_{y \in [0;\, 1]} \max_{x \in [0;\, 1]} K(x,\, y) = \min \left\{ \min_{y \in \left[0;\, -\frac{2a+b}{g}\right]} K(1,\, y),\ \min_{y \in \left[-\frac{2a+b}{g};\, 1\right]} K(x_{\max},\, y) \right\} =$$

$$= \min \left\{ K\left(1,\, y_{\min}^{\langle 1 \rangle}\right),\ \min \left\{ K\left(x_{\max},\, -\frac{2a+b}{g}\right),\ K(x_{\max},\, 1) \right\} \right\} =$$

$$= \min \left\{ K\left(1,\, y_{\min}^{\langle 1 \rangle}\right),\ K\left(x_{\max},\, -\frac{2a+b}{g}\right) \right\} = K\left(1,\, y_{\min}^{\langle 1 \rangle}\right) = a + b - \frac{(g+c)^2}{4h} + k = V_{\text{opt}} \tag{66}$$

is reached on the optimal strategies set (31). Now it is quite clear that the investigated case game solution is the set (36).

Setting up the condition $2h(2a+b)-g(g+c)\leqslant 0$ there appear the point $y_{\min}^{\langle 1\rangle}\geqslant -\dfrac{2a+b}{g}$ and the point $y_{\min}\geqslant -\dfrac{2a+b}{g}$. And as there is the difference

$$y_{\min}-1=\frac{bg-2ac}{4ah-g^2}-1=\frac{bg-2ac-4ah+g^2}{4ah-g^2}=\frac{g(b+g)-2a(c+2h)}{4ah-g^2} \quad (67)$$

then $y_{\min}\leqslant 1$ by $g(b+g)-2a(c+2h)\geqslant 0$.

Case 9. $b+g>0$, $2a+b\geqslant 0$, $2a+b+g\leqslant 0$, $2h(2a+b)-g(g+c)\leqslant 0$, $g(b+g)-2a(c+2h)\geqslant 0$

Having here the double parabolic inequalities (38) and

$$\left\{K\left(x_{\max},-\frac{2a+b}{g}\right)\geqslant K(x_{\max},y_{\min}),\ K(x_{\max},y_{\min})\leqslant K(x_{\max},1)\right\}, \quad (68)$$

the minimum of the function (64) on the segment Y

$$\min_{y\in[0;1]}\max_{x\in[0;1]}K(x,y)=\min\left\{\min_{y\in\left[0;-\frac{2a+b}{g}\right]}K(1,y),\ \min_{y\in\left[-\frac{2a+b}{g};1\right]}K(x_{\max},y)\right\}=$$

$$=\min\left\{\min\left\{K(1,0),\ K\left(1,-\frac{2a+b}{g}\right)\right\},\ K(x_{\max},y_{\min})\right\}=$$

$$=\min\left\{K\left(1,-\frac{2a+b}{g}\right),\ K(x_{\max},y_{\min})\right\}=\min\left\{K\left(x_{\max},-\frac{2a+b}{g}\right),\ K(x_{\max},y_{\min})\right\}=$$

$$=K(x_{\max},y_{\min})=K\left(-\frac{b+gy}{2a},\frac{bg-2ac}{4ah-g^2}\right)=\frac{c(ac-bg)+b^2h}{g^2-4ah}+k=V_{\text{opt}} \quad (69)$$

is reached on the optimal strategies set (40) of the second player. So, the investigated case game solution is the set (44).

Case 10. $b+g>0$, $2a+b\geqslant 0$, $2a+b+g\leqslant 0$, $2h(2a+b)-g(g+c)\leqslant 0$, $g(b+g)-2a(c+2h)<0$

Here appear the points $y_{\min}^{\langle 1\rangle}\geqslant -\dfrac{2a+b}{g}$ and $y_{\min}>1$. It means the double parabolic inequalities (38) and

$$K\left(1,-\frac{2a+b}{g}\right)=K\left(x_{\max},-\frac{2a+b}{g}\right)>K(x_{\max},1)>K(x_{\max},y_{\min}). \quad (70)$$

Then the minimum of the function (64) on the segment Y

$$\min_{y\in[0;1]}\max_{x\in[0;1]}K(x,y)=\min\left\{\min_{y\in\left[0;-\frac{2a+b}{g}\right]}K(1,y),\ \min_{y\in\left[-\frac{2a+b}{g};1\right]}K(x_{\max},y)\right\}=$$

$$=\min\left\{\min\left\{K(1,0),\ K\left(1,-\frac{2a+b}{g}\right)\right\},\ \min\left\{K\left(x_{\max},-\frac{2a+b}{g}\right),\ K(x_{\max},1)\right\}\right\}=$$

$$=\min\left\{K\left(1,-\frac{2a+b}{g}\right),\ K(x_{\max},1)\right\}=\min\left\{K\left(x_{\max},-\frac{2a+b}{g}\right),\ K(x_{\max},1)\right\}=K(x_{\max},1)=$$

$$=K\left(-\frac{b+gy}{2a},1\right)=K\left(-\frac{b+g}{2a},1\right)=\frac{4ah-g^2}{4a}+\frac{2ac-bg}{2a}-\frac{b^2}{4a}+k=c+h-\frac{(b+g)^2}{4a}+k=V_{\text{opt}} \quad (71)$$

is reached on the optimal strategies set (55) of the second player. The corresponding equation (32) now is

$$V_{\text{opt}}=K\left(-\frac{b+g}{2a},1\right)=c+h-\frac{(b+g)^2}{4a}+k=ax^2+bx+gx+c+h+k=K(x,1)=K(x,y_{\text{opt}}); \quad (72)$$

$$ax^2+bx+gx+\frac{(b+g)^2}{4a}=a\left(x+\frac{b+g}{2a}\right)^2=0. \quad (73)$$

The last equation gives the roots $x_1 = x_2 = -\dfrac{b+g}{2a}$ of equation (72). The conditions $b+g>0$ and $2a+b+g\leqslant 0$ mean that the value $-\dfrac{b+g}{2a}\in(0;\,1]$ and the optimal strategies set of the first player

$$X_{\text{opt}} = \{x_1\} = \{x_2\} = \left\{-\frac{b+g}{2a}\right\}. \tag{74}$$

Consequently, the investigated case game solution is the set

$$\mathscr{R} = \left\{\left\{-\frac{b+g}{2a}\right\},\, \{1\},\, c+h-\frac{(b+g)^2}{4a}+k\right\}. \tag{75}$$

Running further, as by the inequality $2a+b+g>0$ there is the inequality $a+b+gy>0\ \ \forall\ y\in[0;\,1]$, then the maximum of the surface (1) on the segment X of the variable x is

$$\max_{x\in[0;\,1]} K(x,\,y) = \max\{K(0,\,y),\, K(1,\,y)\} = K(1,\,y) = a+b+gy+cy+hy^2+k. \tag{76}$$

As $g+c<0$ then here the point $y_{\min}^{\langle 1\rangle}\in(0;\,1]$ by $g+c+2h\geqslant 0$.

Case 11. $b+g>0$, $2a+b\geqslant 0$, $2a+b+g>0$, $g+c+2h\geqslant 0$

As here the point $y_{\min}^{\langle 1\rangle}\in(0;\,1]$ then there is the double parabolic inequality

$$\left\{K(1,\,0)>K\left(1,\,y_{\min}^{\langle 1\rangle}\right),\, K\left(1,\,y_{\min}^{\langle 1\rangle}\right)\leqslant K(1,\,1)\right\}. \tag{77}$$

Then the minimum of the parabola (76) on the segment Y

$$\min_{y\in[0;\,1]}\max_{x\in[0;\,1]} K(x,\,y) = \min_{y\in[0;\,1]} K(1,\,y) = \min_{y\in[0;\,1]}\left(a+b+gy+cy+hy^2+k\right) =$$

$$= K\left(1,\,y_{\min}^{\langle 1\rangle}\right) = a+b-\frac{(g+c)^2}{4h}+k = V_{\text{opt}} \tag{78}$$

is reached on the set (31). Clearly, that the investigated case game solution is the set (36) once again.

Case 12. $b+g>0$, $2a+b\geqslant 0$, $2a+b+g>0$, $g+c+2h<0$

As here the point $y_{\min}^{\langle 1\rangle}>1$ then there is the double parabolic inequality

$$K(1,\,0)>K(1,\,1)>K\left(1,\,y_{\min}^{\langle 1\rangle}\right), \tag{79}$$

meaning that the minimum of the parabola (76) on the segment Y

$$\min_{y\in[0;\,1]}\max_{x\in[0;\,1]} K(x,\,y) = \min_{y\in[0;\,1]} K(1,\,y) = \min\{K(1,\,0),\, K(1,\,1)\} =$$

$$= K(1,\,1) = a+b+g+c+h+k = V_{\text{opt}} \tag{80}$$

is reached on the set (55). The roots of the corresponding equation (32)

$$V_{\text{opt}} = K(1,\,1) = a+b+g+c+h+k = ax^2+bx+gx+c+h+k =$$

$$= a(x-1)\left(x+\frac{a+b+g}{a}\right)+a+b+g+c+h+k = K(x,\,1) = K\left(x,\,y_{\text{opt}}\right) \tag{81}$$

are $x_1=1$ and $x_2=-\dfrac{a+b+g}{a}$. But $2a+b+g>0$ means that $-\dfrac{a+b+g}{a}>1$. Then the optimal strategies set of the first player is (35). Consequently, the investigated case game solution is the set

$$\mathscr{R} = \{\{1\},\,\{1\},\, a+b+g+c+h+k\}. \tag{82}$$

Taking on the factor $2a+b<0$ by $b+g>0$, the kernel (1) maximum is (10), owing to the value $-\dfrac{2a+b}{g}<0$ and the formula (64). With the difference (67) the point $y_{\min}\in(0;\,1]$ by $g(b+g)-2a(c+2h)\geqslant 0$.

Case 13. $b+g>0$, $2a+b<0$, $g(b+g)-2a(c+2h)\geqslant 0$

As here the minimum point $y_{\min}\in(0;\,1]$ then there is the double parabolic inequality

$$\{K(x_{\max},\,0)>K(x_{\max},\,y_{\min}),\, K(x_{\max},\,y_{\min})\leqslant K(x_{\max},\,1)\}, \tag{83}$$

whereupon the minimum of the parabola (10) on the segment Y

$$\min_{y\in[0;\,1]}\max_{x\in[0;\,1]} K(x,\,y) = \min_{y\in(0;\,1]} K(x_{\max},\,y) =$$

$$= K(x_{\max},\,y_{\min}) = K\left(-\frac{b+gy}{2a},\,\frac{bg-2ac}{4ah-g^2}\right) = \frac{c(ac-bg)+b^2h}{g^2-4ah}+k = V_{\mathrm{opt}} \quad (84)$$

is reached on the set (40), and the investigated case game solution is the set (44).

Case 14. $b+g>0$, $2a+b<0$, $g(b+g)-2a(c+2h)<0$

In this relational topology the minimum point $y_{\min}>1$ by the double parabolic inequality

$$K(x_{\max},\,0) > K(x_{\max},\,1) > K(x_{\max},\,y_{\min}). \quad (85)$$

Then the minimum of the parabola (10) on the segment Y

$$\min_{y\in[0;\,1]}\max_{x\in[0;\,1]} K(x,\,y) = \min_{y\in(0;\,1]} K(x_{\max},\,y) =$$

$$= \min\left\{K(x_{\max},\,0),\;K(x_{\max},\,1)\right\} = K(x_{\max},\,1) = K\left(-\frac{b+gy}{2a},\,1\right) = K\left(-\frac{b+g}{2a},\,1\right) =$$

$$= \frac{4ah-g^2}{4a}+\frac{2ac-bg}{2a}-\frac{b^2}{4a}+k = c+h-\frac{(b+g)^2}{4a}+k = V_{\mathrm{opt}} \quad (86)$$

is reached on the optimal strategies set (55) of the second player, whence the optimal strategies set of the first player (74) gives the investigated case game solution in the set (75).

Now if to aggregate and group the investigated 14 cases into the table below, then it is certain, that there are the six unique solution cases, being in the pure strategies, where the second player never applies its minimal pure strategy $y=0$.

Table 1 –The six unique solutions of the specified strictly convex-concave continuous antagonistic game

Relational topologies in the kernel (1) by $a<0$, $b>0$, $g<0$, $c<0$, $h>0$, $k\in\mathbb{R}$	Game solution $\mathscr{R}=\{X_{\mathrm{opt}},\,Y_{\mathrm{opt}},\,V_{\mathrm{opt}}\}$
Case 1. $b+g\leqslant 0$, $2a+b\geqslant 0$, $2h(2a+b)-g(g+c)>0$	$\mathscr{R}=\left\{\{1\},\,\left\{-\frac{g+c}{2h}\right\},\,a+b-\frac{(g+c)^2}{4h}+k\right\}$
Case 8. $b+g>0$, $2a+b\geqslant 0$, $2a+b+g\leqslant 0$, $2h(2a+b)-g(g+c)>0$	
Case 11. $b+g>0$, $2a+b\geqslant 0$, $2a+b+g>0$, $g+c+2h\geqslant 0$	
Case 2. $b+g\leqslant 0$, $2a+b\geqslant 0$, $2h(2a+b)-g(g+c)\leqslant 0$, $2hb-cg>0$	$\mathscr{R}=\left\{\left\{\frac{cg-2bh}{4ah-g^2}\right\},\,\left\{\frac{bg-2ac}{4ah-g^2}\right\},\,\frac{c(ac-bg)+b^2h}{g^2-4ah}+k\right\}$
Case 5. $b+g\leqslant 0$, $2a+b<0$, $2hb-cg>0$	
Case 9. $b+g>0$, $2a+b\geqslant 0$, $2a+b+g\leqslant 0$, $2h(2a+b)-g(g+c)\leqslant 0$, $g(b+g)-2a(c+2h)\geqslant 0$	
Case 13. $b+g>0$, $2a+b<0$, $g(b+g)-2a(c+2h)\geqslant 0$	
Case 3. $b+g\leqslant 0$, $2a+b\geqslant 0$, $2h(2a+b)-g(g+c)\leqslant 0$, $2hb-cg\leqslant 0$, $c+2h\geqslant 0$	$\mathscr{R}=\left\{\{0\},\,\left\{-\frac{c}{2h}\right\},\,k-\frac{c^2}{4h}\right\}$
Case 6. $b+g\leqslant 0$, $2a+b<0$, $2hb-cg\leqslant 0$, $c+2h\geqslant 0$	
Case 4. $b+g\leqslant 0$, $2a+b\geqslant 0$, $2h(2a+b)-g(g+c)\leqslant 0$, $2hb-cg\leqslant 0$, $c+2h<0$	$\mathscr{R}=\{\{0\},\,\{1\},\,c+h+k\}$
Case 7. $b+g\leqslant 0$, $2a+b<0$, $2hb-cg\leqslant 0$, $c+2h<0$	
Case 10. $b+g>0$, $2a+b\geqslant 0$, $2a+b+g\leqslant 0$, $2h(2a+b)-g(g+c)\leqslant 0$, $g(b+g)-2a(c+2h)<0$	$\mathscr{R}=\left\{\left\{-\frac{b+g}{2a}\right\},\,\{1\},\,c+h-\frac{(b+g)^2}{4a}+k\right\}$
Case 14. $b+g>0$, $2a+b<0$, $g(b+g)-2a(c+2h)<0$	
Case 12. $b+g>0$, $2a+b\geqslant 0$, $2a+b+g>0$, $g+c+2h<0$	$\mathscr{R}=\{\{1\},\,\{1\},\,a+b+g+c+h+k\}$

The formulated above theorem has been proved.

The investigated game and the proved theorem conclusion

The proved theorem may be used with the table 1 for the fast decision-making in conflict systems, which may be modeled as the continuous antagonistic games [1, 3, 5, 6, 8]. In modeling processes of machine-building aggregates run-in time selection the second player (projector) should hold its pure optimal strategy as the relative run-in time. This gives at once the pragmatically single solution on how long to take a machine-building aggregate at the run-in time. The future investigations should be directed towards proving theorems for the continuous antagonistic games, which are solved in the mixed strategies.

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