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Mathematical models of individualised learning based on decision theory

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Abstract. The study provided theoretical substantiation and development of a system of mathematical models for the individualisation of the educational process based on the integration of decision theory methods. The developed system of mathematical models is based on a metamodel that combines four mathematical paradigms through an interaction matrix, the elements of which are determined by the function of cognitive compatibility, temporal consistency and interaction efficiency. The introduction of the method of optimising partial trajectories, based on recursive updating of model parameters through the analysis of intermediate results, increased the accuracy of parameter settings and ensured smooth adaptation to the individual learning rate. The developed modification of the Bellman equation with the function of the complexity of the learning material made it possible to formalise the process of optimising long-term learning strategies by addressing individual cognitive characteristics. The analysis of the stochastic nature of the learning process through an extended transition matrix was used to mathematically describe the processes of forgetting and repeating the material using a system of differential equations with time-dependent coefficients that account for the intensity of learning and individual memory characteristics. The study of collaborative learning mechanisms using the game-theoretic approach revealed the synergistic effects of group learning through nonlinear functions of interaction between participants in the educational process and has allowed the development of methods for forming optimal learning groups based on individual goals. The proposed system of multidimensional evaluation, implemented through a composite objective function, covers a wide range of indicators from basic knowledge acquisition to the development of higher-order metacognitive skills, including cognitive, metacognitive and motivational components, which provides a reliable tool for assessing the stability of learning trajectories and determining the level of adaptability of the system to individual characteristics of students

Keywords: adaptive educational systems; Bayesian optimisation; Bellman function; Markov processes; game-theoretic approach; cognitive trajectories

Introduction

Modern educational space features a rapid transition from unified approaches to individualised learning, driven by the growing need to effectively adapt the educational process to the individual characteristics of each student. Traditional teaching methods often do not account for the diversity of cognitive styles, learning styles and prior experience of students, which leads to a decrease in the effectiveness of the educational process. Mathematical modelling of individualised learning processes is of particular importance in the context of the development of digital educational technologies and decision support systems. The use of mathematical models formalises and optimises the process of adapting educational content, creating personalised educational trajectories and predicting learning outcomes. The 2019-2024 research significantly expanded the understanding of multicriteria approaches in the educational context. H. Taherdoost & M. Madanchian (2023) developed a comprehensive approach to the application of Multi-Criteria Decision-Making (MCDM) methods. This is a group of methods for decision-making that account for multiple criteria or alternatives. Detailing their potential for creating adaptive educational systems and proposing a

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methodological framework for decision-support systems in education. In the development of this direction, I. Canco *et al.* (2021) presented a practical implementation of the Analytical Hierarchy Process (AHP) method, demonstrating its effectiveness in shaping individual educational trajectories and developing a system of criteria for assessing the quality of the educational process. A fundamentally new perspective of decision-making mechanisms was proposed in a study by J.C. Peterson *et al.* (2021), which, through large-scale experiments and the use of machine learning methods, identified fundamental patterns in decision-making processes, which can be used as the basis for the development of predictive models with greater accuracy in education.

The introduction of Bayesian networks and adaptive models created new horizons in the field of automated learning assessment. W. Xing *et al*. (2021) developed an innovative model for assessing students' engineering projects that not only accounts for multiple success factors but also adapts to the individual progress of each student, ensuring objective and personalised assessment. A significant step forward was made by L.G. Eglington & P.I. Pavlik (2023) proposed methods of rapid learning optimisation that accounts for the individual pace of learning, cognitive characteristics and previous experience of students, which can be used for dynamic adjustment of task complexity and learning pace. A comprehensive view of the future of personalised education was presented in a study by S. Maghsudi *et al.* (2021), which not only outlined promising areas of development but also proposed specific mechanisms for integration of artificial intelligence into the educational process, highlighting the need for a balance between automation and preservation of the human factor in educational process.

Recent advances in the field of Bayesian optimisation significantly expanded the possibilities of mathematical modelling of educational processes. X. Wang *et al.* (2023) presented a thorough analysis of modern Bayesian optimisation methods, systematising existing approaches and outlining promising areas for their application in the educational context, especially for optimising the parameters of educational systems and predicting educational outcomes. The innovative concept of inexact Bayesian optimisation was proposed by J. Rodemann & T. Augustin (2024), which improves efficiency in uncertainty in the educational process, addressing multiple factors of influence and their interdependence. A significant breakthrough in the development of predictive models was made by P. Jiang & X. Wang (2020), they have developed a cognitive diagnostic method that not only predicts student performance but also identifies specific areas that require additional attention, which allows for proactive adaptation of the learning process.

A comprehensive analysis of existing research reveals both significant achievements and significant gaps in existing approaches to the mathematical modelling of individualised learning. S. Minn (2022), exploring the possibilities of artificial intelligence in knowledge assessment, emphasised the need to create integrated systems that combine different methodological approaches. In particular, the mechanisms of interaction of different mathematical models within a single decision-making system, methods of validation and evaluation of the effectiveness of such models in a real educational environment, as well as ways to adapt the models to different educational contexts and cultures remain insufficiently researched. Particular attention should be devoted to the development of methods that address not only the cognitive aspects of learning but also the emotional and motivational dimensions, ensuring a genuinely holistic approach to individualising the educational process. While existing research has laid a solid foundation for further advancement in this area, there remains a pressing need for systematic integration and practical validation of these methods across a wide range of educational contexts.

Research objective: theoretical substantiation and development of a system of mathematical models of individualised learning based on the integration of Bayesian optimisation, dynamic programming, Markov processes and game theory to create a methodological basis for decision-making in the educational process. Tasks of the study were:

- To analyse the theoretical foundations and methodological approaches to the construction of mathematical models of individualised learning, in particular models of Bayesian optimisation, dynamic programming, Markov processes and game theory.
- To develop a theoretical justification for the integration of various mathematical models into a single decision-making system for individualised learning, addressing their interaction and limitations.
- To analyse the theoretical aspects of evaluating the effectiveness of the proposed mathematical models and develop methodological foundations for their practical application in the context of predicting educational outcomes.

Materials and Methods

The study analysed four main mathematical approaches – Bayesian optimisation, dynamic programming, Markov processes and game theory – as the basic components of an individualised learning system. Within the framework of these approaches, a multicomponent fitness function was investigated, which reflects a complex vector of characteristics of learning materials, including the complexity of the content, the time required for learning, the way the material is presented, the level of interactivity and the pre-requisites for students' knowledge. The analysis of these characteristics was used to explore methods for optimising partial learning trajectories based on recursive updating of model parameters following intermediate learning outcomes. In the study of long-term learning strategies, the method of dynamic programming was applied the modified Bellman equation was analysed in the educational context. The analysis included the study of the dependencies between the function of the complexity of the educational material and the processes of optimising individual learning trajectories. The mechanisms of adaptation of learning parameters to individual characteristics of students, including the pace of learning, cognitive styles and previous experience, are investigated. This determined the principles of building long-term strategies for individualised learning, considering the dynamics of the learning process and the peculiarities of material perception.

The use of Markov processes addressed the stochastic nature of learning through the analysis of an extended transition matrix that considers conditional probabilities for different learning actions and states. When studying the processes of forgetting and repeating material, a system of differential equations with time-dependent coefficients reflecting changes in knowledge acquisition was analysed. This analysis examined the relationships between the intensity of learning, the complexity of the material and the individual characteristics of students' cognitive processes, including memory characteristics, information processing speed and the stability of the acquired knowledge.

The study of game theory analysed theoretical and game models of collaborative learning, in particular, the functions of forming learning coalitions and their impact on the educational process. The synergistic effects of students' interaction were investigated through the analysis of nonlinear functions that reflect the individual characteristics of participants in the educational process. Based on the game-theoretic approach, the principles of group learning assessment and criteria for the formation of study groups were analysed, addressing individual goals, level of training and peculiarities of interaction between participants in the educational process. In the study of the interaction of mathematical models, their main properties were analysed: computational complexity, stability of functioning and adaptability to different educational contexts. The methods of optimising model parameters and mechanisms of their integration into a single decision-making system were investigated. The possibilities of combining Bayesian optimisation, dynamic programming, Markov processes and game theory to provide a comprehensive approach to individualised learning are analysed. The conditions and limitations of interaction between different mathematical models in the context of the educational process are determined.

The study analysed the systems of multicriteria evaluation of mathematical models through a composite objective function that considers the cognitive, metacognitive and motivational components of the educational process. The criteria for assessing the stability of learning trajectories and the level of adaptability of the system to the individual characteristics of students were investigated. The methodological approaches to evaluating the effectiveness of the models both in the short term (assessment of basic knowledge acquisition, intermediate results) and the long term (development of metacognitive skills, sustainability of acquired knowledge) were determined. The results of the theoretical study were systematised in two analytical tables, which present a comparative analysis of the theoretical characteristics of mathematical models and criteria for assessing their effectiveness in the context of individualised learning.

Results

Theoretical and methodological foundationsof mathematical modelling of individualised learning

The theoretical study of mathematical models of individualised learning reveals the potential of an integrated approach that combines four mathematical paradigms, each of which is unique in the overall architecture of the system. Bayesian optimisation in the mathematical modelling system provides adaptive adjustment of the learning content parameters through a multicomponent fitness function $f(x)$, where x represents a multidimensional vector of characteristics of learning materials, including content complexity, learning time, type of material presentation, and other key parameters. The implementation of the partial trajectory optimisation method is based on recursive updating of the model parameters through the analysis of intermediate learning outcomes. This approach improves the accuracy of parameter settings, ensuring adaptation to the individual pace of learning for each student. The theoretical analysis of mathematical models of this class demonstrates their potential for individualising learning when working with groups of students with different learning styles (Zhang *et al.*, 2023). The dynamic programming system is developed with a focus on optimising long-term learning strategies through a modified Bellman equation adapted to the specifics of the educational context. The mathematical formalisation is based on the principle of optimality.

$$V\times(s) = max[R(s, a) + \gamma\Sigma P(s'|s, a)\, V^*(s')], \tag{1}$$

where: $V\times(s)$ – optimal value function for state s; $R(s, a)$ – direct reward for action a in state s; γ – discount factor for future rewards; $\gamma \in [0,1]$, $P(s'|s, a)$ – probability of transition to state s' from state s when performing action a.

To account for the specifics of the educational process, an additional function of the complexity of the educational material $C(s, a)$ is introduced, which modifies the basic equation to the following form:

$$V^*(s) = max[R(s, a)/C(s, a) + \gamma\Sigma P(s'|s, a) V^*(s')], \tag{2}$$

where: $C(s, a)$ – function of the complexity of the learning material, the rest of the notation is similar to formula (1).

The value function $V(s)$ provides a comprehensive assessment of the optimality of learning trajectories through multivariate analysis, which is implemented by the parameter vector θ:

$$\theta = \{\theta 1, \ldots, \theta n\}, \tag{3}$$

where: θi – components that correspond to different aspects of learning (cognitive development, metacognitive skills, level of learning).

Each parameter θi is optimised according to a function:

$$\theta i^* = argmax\{E[V(s|\theta i)] - \lambda R(\theta i)\}, \tag{4}$$

where: θi^* – optimal value of the parameter; $E[V(s|\theta i)]$ – expected value of the state at parameter θi; $\lambda R(\theta i)$ – regularisation term to prevent overlearning.

Long-term effects are modelled using a composite forecasting function:

$$F(s, t) = \sum \alpha i fi(s, t) + \beta \sum\sum wij(fi(s, t) \cdot fj(s, t)), \tag{5}$$

where: αi – weighting coefficients of individual components; wij – interaction coefficients between different aspects of learning; $fi(s, t)$ – components of the prediction function; β – overall interaction coefficient.

The integration of Markov processes into the system addresses the stochastic nature of the learning process through the transition matrix $P = [pij]$, supplemented by conditional probabilities of transitions at different learning actions:

$$P(s'|s, a) = P(s'|s) \cdot P(a|s, s'), \tag{6}$$

where: $P(s'|s)$ – probability of transition from state s' to state s'; $P(a|s, s')$ – conditional probability of action a during the transition between states.

The developed variational approach for analysing time series of training data is based on functional optimisation:

$$L'(\theta) = L(\theta) - \lambda \sum ||\nabla \theta L(\theta)||^2, \tag{7}$$

where – $L'(\theta) = E[log\ p(x|\theta)] - DKL(q(z|x)||p(z|x,\theta))$ basic functionality; λ – regularisation coefficient; $\nabla \theta L(\theta)$ – functionality gradient.

The stability of learning trajectories is analysed using a range of Lyapunov indicators:

$$\lambda i = lim(t \rightarrow \infty)\ (1/t) log(||\delta xi(t)||/||\delta xi(0)||), \tag{8}$$

where: λi – Lyapunov exponent; $\delta xi(t)$ – trajectory deviation at time t; $\delta xi(0)$ – initial deviation.

Such an extended theoretical framework ensures not only the mathematical rigour of the model but also its practical applicability in various educational contexts, guaranteeing robustness under different initial conditions and external influences. The model demonstrates effectiveness in nonlinear learning trajectories and complex conceptual structures, which is confirmed by the theoretical analysis of its properties. The theoretical and game component of the system is implemented through a multi-level model of strategic interaction of participants in the educational process. The key focus is on modelling collaborative learning and competitive interaction. In the mathematical formalisation, each participant in the educational process acts as a player with a personal set of strategies:

$$Si = \{si1, \ldots, sin\}, \tag{9}$$

where: Si – set of strategies of the i-th player; sij – j-th strategy of the i-th player.

To improve the effectiveness of learning, mechanisms for forming coalitions were developed to group students based on common goals and interests:

$$C = \{c1, \ldots, ck\}, \tag{10}$$

where: C – set of coalitions; ci – i-th coalition; k – number of coalitions.

To assess the success of the learning process, the system uses a multi-criteria evaluation function that accounts for various aspects of learning:

$$F(x) = \sum wi \cdot fi(x), \tag{11}$$

where: wi – weighting coefficients; $fi(x)$ – functions that reflect different aspects of the learning process; x – vector of learning process parameters.

When students interact, synergistic effects arise, which are modelled through nonlinear interaction functions:

$$g(xi,xj) = \alpha \cdot xi \cdot xj + \beta \cdot (xi + xj), \tag{12}$$

where: xi, xj – student characteristics; α, β – interaction coefficients.

To create a holistic system of adaptive learning, the four main paradigms are integrated through a metamodel:

$$M = \{BO, DP, MP, GT\}, \tag{13}$$

where: BO – Bayesian optimisation; DP – dynamic programming; MP – Markov processes; GT – game theory.

The analysis of the theoretical characteristics of the developed mathematical models can be used to systematise their key parameters and features of application in the context of individualised learning (Table 1).

Table 1. Theoretical characteristics of mathematical models of individualised learning

Evaluation criterion	BO	DP	MP	GT
Predictive power	High	Average	Average	Low
Theoretical complexity	O(n log n)	O(n²)	O(n²)	O(n^k)
Flexibility of the model	8.5	6.8	8.2	6.4

Table 1. Continued

Evaluation criterion	BO	DP	MP	GT
Theoretical scalability	High	Limited	High	Limited
Mathematical complexity	Moderate	High	Moderate	High

Notes: BO – Bayesian optimisation; DP – dynamic programming; MP – Markov processes; GT – game theory. The flexibility of the model was assessed on a theoretical 10-point scale based on the possibility of adaptation to different educational contexts

Source: compiled by the author based on H. Wu & F. Noé (2020), Q. Cappart *et al*. (2021), D.P. Bertsekas (2022), D. Zhang *et al*. (2023), S. Tu *et al*. (2024)

The table presents a comparative assessment of four mathematical models in the context of individualised learning. Bayesian optimisation (*BO*) stands out for its high predictive power, theoretical scalability, and the highest level of flexibility. Dynamic programming (*DP*) and Markov processes (*MP*) have moderate predictive power but are distinguished by their high mathematical complexity. Game theory (*GT*), despite its low predictive power, remains valuable for analysing complex systems due to its applicability to multifactor scenarios. The theoretical analysis of the developed system of mathematical models reveals their potential for further development of the theory of individualised learning. Each of the presented models has advantages in the context of mathematical modelling of educational processes.

An integrated system of mathematical decision-making models

The integrated system of mathematical models of individualised learning is based on the metamodel $M = \{BO, DP, MP, GT\}$ (where *BO* stands for Bayesian optimisation, *DP* for dynamic programming, *MP* for Markov processes, *GT* for game theory, which together form a comprehensive system for modelling individualised learning). Architecturally, the system implements the principle of deep integration through the interaction matrix $I = [iij]$, whose elements are determined by the function:

$$iij = \alpha \cdot cij + \beta \cdot tij + \gamma \cdot eij, \tag{14}$$

where: α – weighting factor of cognitive compatibility; β – weighting factor of temporal consistency; γ – weighting factor of interaction efficiency; *cij* – indicator of cognitive compatibility of models; *tij* – indicator of temporal consistency; *eij* – indicator of interaction efficiency.

The fundamental basis of the architecture of the integrated system is the principle of adaptive interaction of components, which ensures flexible adjustment of learning parameters following the individual characteristics of students and the dynamics of the educational process (Bertsekas, 2022). The system of intermodule interaction is described by a set of differential equations:

$$dK/dt = FK(K,L) + GK(I), \tag{15}$$

where: K – vector of student's knowledge; L – vector of educational influences; *FK, FL* – developmental functions; *GK, GL* – functions of intermodule interaction; t – time.

The peculiarity of the developed system is its ability to self-adapt through feedback mechanisms, which allows for optimising the learning trajectory in real-time (Wu & Noé, 2020). The Bayesian component of the system interacts with the dynamic programming component through a transfer function:

$$T(BO \rightarrow DP) = \int P(\theta|D)V(s|\theta)d\theta, \tag{16}$$

where: $P(\theta|D)$ – posterior distribution of the model parameters given the available data D; $V(s|\theta)$ – value function of the state s given the parameters θ; D – training data set.

The development of mechanisms for interaction between the system components is based on the principle of synergistic enhancement of learning effects. The integration of Bayesian optimisation with dynamic programming can be used to combine the benefits of probabilistic modelling of student knowledge with the optimisation of long-term learning strategies. In this case, the Bayesian component provides an accurate assessment of the current state of knowledge, and dynamic programming uses these estimates to build optimal learning trajectories (Tu *et al.*, 2024). These interactions are formalised through an extended transfer function:

$$T'(BO \rightarrow DP) = \iint P(\theta|D)V(s|\theta)K(s,s')d\theta ds, \tag{17}$$

where: $P(\theta|D)$ – posterior distribution of the model parameters; $V(s|\theta)$ – state value function; $K(s, s')$ – kernel of transition between knowledge states; D – training data.

An in-depth analysis of the interaction of the system components determined the need to address the temporal aspects of learning. Each transition between states of knowledge is characterised not only by the probability of success but also by the time required to learn the material. This feature is reflected in a modified transition matrix that addresses the time characteristics of the learning process:

$$P(t) = P0 \times exp(At), \tag{18}$$

where: $P0$ – initial transition matrix; A – time evolution generator; t – training time.

Individual student characteristics are addressed through an adaptive system of weighting coefficients that is dynamically adjusted during the learning process. This approach allows the system to automatically determine the most effective presentation strategies for each learner (Zhang *et al.,* 2023). Mathematically, this is expressed as an optimisation problem:

$$W^* = \text{argmin} \sum||yi - f(xi,W)||^2 + \lambda R(W), \tag{19}$$

where: W^* – optimal set of weights; yi – target learning indicators; $f(xi, W)$ – predicted results; $R(W)$ – regularisation term; λ – regularisation coefficient.

An important aspect of an integrated system is the coordination mechanisms between different models, which ensure that all components work in a coordinated manner. The coordination mechanism is implemented through a multi-level decision-making system, where each level is responsible for a specific aspect of the learning process. Mathematically, this is described by the hierarchical structure of decision-making functions:

$$D(s) = H(F1(s), F2(s),\ldots,Fn(s)), \tag{20}$$

where: $D(s)$ – final solution of the system; $Fi(s)$ – solution of the i-th level; H – solution aggregation function; s – current state of the learning process.

The analysis of the processes of adaptation of educational content demonstrated the need to introduce mechanisms for dynamic optimisation of the complexity of the material. The system uses a composite difficulty assessment function that accounts for multiple characteristics of the learning material and individual student characteristics:

$$C(m,u) = \beta 1c1(m) + \beta 2c2(u) + \beta 3c3(m,u), \tag{21}$$

where: m – characteristics of the educational material; u – characteristics of the student; ci – components of the complexity assessment; βi – weighting coefficients.

The system emphasises modelling the processes of forgetting and repeating material. Based on the extended forgetting curve, an adaptive algorithm for scheduling repetitions that addresses the individual characteristics of a student's memory has been developed (Wu & Noé, 2020). This process is described by a differential equation:

$$dR/dt = -\alpha(t)R + \beta(t)L + \gamma(t)S, \tag{22}$$

where: R – level of knowledge retention; L – intensity of learning; S – complexity of the material; $\alpha(t)$, $\beta(t)$, $\gamma(t)$ – time-dependent coefficients.

The proposed system ensures the stability of the learning process even with significant variations in input parameters. This is achieved through mechanisms of automatic correction of model parameters based on the analysis of current training results. The stability of the system is characterised by a range of Lyapunov indicators, which assess its sensitivity to perturbations in the initial conditions and external influences (Cappart *et al.*, 2021).

The developed criteria for optimising the learning process account for multiple aspects of individual learning through a comprehensive objective function that reflects both immediate learning outcomes and long-term educational goals. The mathematical formalisation of this function includes components of cognitive development, metacognitive skills and motivational factors:

$$Q(x) = \sum wiqi(x) + \sum\sum vijqi(x)qj(x), \tag{23}$$

where: $qi(x)$ – individual criteria of learning quality; wi – weighting coefficients of criteria; vij – coefficients of interaction between criteria; x – vector of parameters of the learning process.

The study of the dynamics of the educational process revealed the need to address nonlinear effects in the acquisition of knowledge. To this end, an adaptive algorithm for adjusting the complexity of learning tasks based on the analysis of the student's current state of knowledge and learning history was developed. This algorithm employed a recurrent neural architecture to predict the optimal level of difficulty:

$$h(t) = \sigma(Wh \cdot h(t-1) + Wx \cdot x(t) + b), \tag{24}$$

where: $h(t)$ – hidden state of the model; $x(t)$ – input data on the learning process; Wh, Wx – weighting matrices; b – displacement vector; σ – activation function.

The analysis of the processes of forming a deep understanding of the material demonstrated the need to introduce mechanisms for identifying and eliminating knowledge gaps. To this end, a knowledge diagnostic system based on Bayesian networks and accounting for the relationships between different concepts of the subject area has been developed:

$$P(K|E) = \prod P(Ki|Pa(Ki)) \cdot P(Ej|Ki), \tag{25}$$

where: K – vector of knowledge state variables; E – vector of observed data; $Pa(Ki)$ – set of parent nodes for Ki; $P(Ki|Pa(Ki))$ – conditional probability of knowledge of concepts; $P(Ej|Ki)$ – probability of observations with the given knowledge.

The integration mechanisms of the system ensure adaptive adjustment of the learning process through continuous analysis and optimisation of a set of parameters. At the same time, each component of the system functions as part of a single whole, providing a synergistic effect in achieving educational goals. The greatest efficiency is achieved by dynamically balancing different learning strategies when the system automatically selects the optimal ratio between the depth of study and the speed of progression through the curriculum. This approach adapts the system to the individual characteristics of each learner, considering their cognitive abilities, previous experience and current level of understanding of the material.

The developed algorithms for adapting educational content are based on the principles of deep learning and cognitive psychology. The system constantly analyses the patterns of interaction between students and learning material, identifies the characteristics of their learning style and automatically adjusts the parameters of information presentation. The system addresses not only explicit performance indicators, but also hidden indicators of understanding, such as time spent on tasks, error patterns, and the nature of requests for help. This multi-level analysis system makes it possible to generate accurate predictions about the most effective learning strategies for each case.

An important aspect of the integrated system is the self-learning algorithm-improvement method based on the experience gained. By analysing large amounts of data on learning trajectories, the system identifies hidden patterns in the learning process and automatically optimises its parameters to improve learning efficiency. At the same time, the system maintains a high level of interpretability of its decisions, enabling teachers to analyse the logic of its operation and adjust its functioning if necessary. This system architecture strikes an optimal balance between automating the learning process and maintaining control by the teaching staff, creating an effective environment for individualised learning.

Methodological principles of efficiency assessment and practical application

The implementation of a multidimensional assessment system for mathematical models of individualised learning is based on a comprehensive analysis of educational outcomes. The system of criteria covers a wide range of indicators: from basic knowledge acquisition to higher-order metacognitive skills and the ability to apply knowledge in practice. The multidimensional assessment model proposed in educational effectiveness research involves simultaneous tracking of students' academic performance, development of their metacognitive abilities and level of learning motivation (De Maeyer *et al.*, 2010). The development of mathematical modelling skills among students is prioritised, which creates the basis for a deeper understanding of the material and the development of analytical thinking (On evaluating curricular..., 2004). The monitoring system includes technologies for continuously collecting and analysing data on individual learning trajectories, which allows for tracking the dynamics of key competencies and making the necessary adjustments to the learning process.

The effective use of mathematical models in the educational process requires the creation of a comprehensive system of teacher training. Specialised professional development programmes should cover both the theoretical foundations of mathematical modelling in education and the practical aspects of using modelling to individualise learning. Training programmes should include intensive practical sessions on the development and adaptation of learning materials, as well as methods for evaluating the effectiveness of different models in specific educational contexts (Aydogan Yenmez *et al.*, 2017). The integration of an engineering approach to mathematical modelling into the educational process opens up new opportunities for the development of students' analytical and design skills (Lyon & Magana, 2020). At the same time, it is necessary to strike a balance between technological innovation and preserving the human factor in the educational process, where the role of the teacher is transformed from a simple transmitter of knowledge to a facilitator of individual student development. Mathematical models in the educational process transform approaches to assessing students' knowledge and skills. Assessment goes beyond traditional tests and begins to include an analysis of student's ability to create personal models to solve practical problems. This approach allows for a deeper understanding of the level of learning and identifies gaps in understanding of basic concepts. The development of tasks for mathematical modelling requires teachers to have a deep understanding of both the subject area and methods of assessing student work (Dogan, 2020). At the same time, it is necessary to develop students' self-assessment and peer assessment skills, which contributes to a deeper understanding of modelling processes.

Adaptation of learning materials to the individual needs of students requires a flexible approach to the organisation of the educational process. Mathematical models can be used to create dynamic learning trajectories that are automatically adjusted based on current results and the pace of learning. Modelling in engineering education shows that an individualised approach significantly increases student motivation and engagement in the learning process (Lyon & Magana, 2020). Analysis of practical implementation experience shows the need to create a bank of tasks of different levels of complexity and thematic focus to ensure effective differentiation of learning.

The development of digital technologies expands the possibilities for introducing complex mathematical models into everyday pedagogical practice. The integration of mathematical modelling into the learning process is becoming a tool for developing students' critical thinking and analytical skills. It is necessary to ensure a gradual transition from simple models to more complex ones, which allows students to build their competencies at a natural pace (Bora & Ahmed, 2019). The technological infrastructure should support different learning formats, from individual work to group projects while ensuring continuous data collection to analyse the effectiveness of the learning process. The introduction of mathematical models into the learning process requires the creation of an adaptive learning environment. A key element of such an environment is an automated decision support system that helps teachers determine the best learning strategies for each student. The analysis of the results of modelling educational processes shows that the effectiveness of individualised learning depends not only on the accuracy of mathematical models but also on the quality of their integration into pedagogical practice. Mathematical modelling expands opportunities for the development of student's creative thinking and their ability to solve complex problems independently. At the same time, it is necessary to strike a balance between technological innovations and maintaining a lively dialogue between teacher and student.

Evaluation of the long-term effectiveness of mathematical models requires the development of a monitoring system that tracks not only the immediate learning outcomes but also the development of students' metacognitive skills. A multidimensional approach to assessment should address students' ability to apply their knowledge in new contexts, their ability to analyse their own mistakes and adjust their learning strategies. Analysis of learning achievements through the prism of mathematical modelling reveals new aspects of understanding the processes of knowledge acquisition and skill development (De Maeyer *et al.*, 2010). This approach identifies hidden patterns in the learning process and develops more effective strategies for individualising learning. The development of a decision support system in education requires constant updating and improvement of mathematical models in line with new pedagogical research and technological capabilities. At the same time, it is important to maintain a focus on developing students' critical thinking and creativity without turning the learning process into a mechanical execution of algorithms. Mathematical modelling should become a tool for developing students' ability to learn independently and conduct research, forming the basis for their further professional development.

The introduction of mathematical models into the educational process opens new horizons for the development of education, transforming traditional approaches to teaching and assessment. The development and application of these models create a multidimensional space of possibilities where each student can follow their educational trajectory. At the same time, the key success factor is not only the technological complexity of the models but also their ability to adapt to the individual characteristics of each student, ensuring an optimal balance between challenges and support in the learning process. The analysis of the characteristics of the effectiveness of these models reveals their potential for creating a truly personalised educational environment where technology and pedagogical skills work in harmony. Summarising the results of the study of methodological foundations for assessing the effectiveness of mathematical models of individualised learning systematised the key characteristics of their practical application in the educational process (Table 2).

Table 2. Characteristics of the effectiveness of mathematical models in individualised learning

Evaluation criterion	Short-term perspective	Long-term perspective
Knowledge assessment	Testing, practical tasks, projects	Analysis of metacognitive skills, self-learning capabilities
Adaptability of learning	Adjustment of task complexity and learning pace	Formation of individual educational trajectories
Monitoring progress	Daily analysis of results, feedback	Tracking the dynamics of competence development
Teacher support	Automation of routine tasks, analytics	Professional development and methodological support
Technology integration	Basic digital tools, LMS	Integrated adaptive systems, AI support

Notes: LMS – Learning Management System; AI – Artificial Intelligence. Knowledge assessment includes both formal and informal methods of evaluation. Adaptability of learning involves automatic adjustment of the learning process parameters

Source: compiled by the author based On evaluating curricular... (2004), S. De Maeyer *et al*. (2010), A. Aydogan Yenmez *et al*. (2017), J.A. Lyon & A.J. Magana (2020)

The introduction of mathematical models of individualised learning creates a foundation for the transformation of the educational process towards greater personalisation and efficiency. The developed methodological framework for evaluating the effectiveness and recommendations for the practical application of the models provide the basis for further development and improvement of the individualised learning system, opening new opportunities for improving the quality of education through the integration of technological innovations and pedagogical experience.

Discussion

The theoretical study of the system of mathematical models of individualised learning, which integrates Bayesian optimisation, dynamic programming, Markov processes and game theory, reveals significant potential in improving the efficiency of the learning process. The analysis of mathematical models has revealed a theoretical increase in the efficiency of individualised learning compared to traditional methods, especially in groups of students with different learning styles. The study of the method of optimising partial trajectories based on recursive updating of model parameters indicates the possibility of increasing the accuracy of parameter settings and ensuring smoother adaptation to the individual pace of learning.

In the context of the dynamic adaptation of the learning process, a comparison of the theoretical results with the study by L. Tetzlaff *et al.* (2021) is noteworthy. Their concept of the dynamic structure of personalised education resonates with the considered system of Bayesian optimisation of learning parameters, but the theoretical model proposed in the current study demonstrates a deeper integration of feedback mechanisms through the analysis of a multicomponent fitness function. At the same time, the results of a study by L. Tetzlaff *et al.* on the temporal aspects of learning process adaptation confirmed the importance of the mechanism of dynamic adjustment of learning parameters through a modified transition matrix considered in this study. The cognitive aspects of the decision-making process in the learning environment were highlighted by J.C. Peterson *et al.* (2021). Their findings on the use of large-scale experiments and machine learning to uncover human decision-making mechanisms confirmed the theoretical effectiveness of using Markov processes in modelling educational decisions. However, the model considered in the current study offers a more specific adaptation to the educational context through the study of the modified Bellman equation and the learning material complexity function, which theoretically allows for a more accurate consideration of the peculiarities of individual perception of educational content.

A significant contribution to understanding the mechanisms of fine-tuning educational parameters was made by H. Luan & C.-C. Tsai (2021). Their analysis of the application of machine learning to precision education confirms the theoretical results obtained in the current study on improving the effectiveness of individualised learning. However, the system under consideration demonstrates a more comprehensive theoretical approach through the integration of four mathematical paradigms and the study of coalition formation mechanisms to group students into groups with common goals and interests. The comparison of the theoretical results with the study by L. Zhang *et al.* (2020) is particularly noteworthy. Their conclusions regarding the need for a systematic approach to personalisation are confirmed in the model under consideration, but the proposed approach provides more specific mathematical mechanisms for implementation through a theoretical analysis of the integration of Bayesian optimisation, dynamic programming, Markov processes, and game theory. At the same time, the results of L. Zhang *et al.* on the importance of taking into account the social aspects of learning emphasised the feasibility of the game-theoretic component of the system studied in the current work. A relevant aspect of the theoretical study of mathematical models of individualised learning is the consideration of social and cognitive theories and their impact on the learning process. S. Chuang (2021) emphasised the importance of continuous adult development in the study on the application of constructivist learning theory and social learning theory. The results of this study correlate with the theoretical and game components of the system considered in the current study, especially in the context of coalition formation and the modelling of collaborative learning. However, the theoretical model presented in this study offers a more formalised approach through mathematical modelling of social interactions in the learning process.

The theoretical analysis of mathematical models of individualised learning revealed the prospects of using the modified Bellman equation to optimise learning trajectories, which incorporates a specific function of the complexity of the learning material C (s, a). This modification addresses not only the direct reward for learning actions but also the individual cognitive characteristics of each student's perception of the material. Integration of Markov processes through the transition matrix P = [pij], supplemented by conditional probabilities of transitions at different learning actions, provides an opportunity to model the stochastic nature of the educational process. In this context, the study by M.A.K. Peters (2022) on decision-making confidence in the educational environment is of particular interest. The analysis of cognitive decision-making mechanisms is consistent with the modified Bellman equation considered in the current study and its adaptation to the educational context. However, the theoretical model under consideration offers a more comprehensive mathematical framework through the integration of Markov processes and Bayesian optimisation, which formalises the stochastic nature of the learning process and the mechanisms of adaptation to individual learners.

Of considerable interest in the context of artificial intelligence in education is the study by F. Ouyang & P. Jiao (2021), which identifies three paradigms of AI application in education: instrumental (AI as a learning support tool), pedagogical (AI as an adaptive tutor), and transformational (AI as an agent of change in the educational process). Their conclusions regarding the need to integrate different approaches are reflected in the theoretical model under consideration, which combines four mathematical paradigms: Bayesian optimisation for adaptive adjustment of learning content parameters, dynamic programming for optimisation of long-term learning strategies, Markov processes for modelling the stochastic nature of the educational process, and game theory for formalising the interaction of learners. However, the system proposed in the current study demonstrates a deeper mathematical formalisation of the processes of individualisation of learning through the introduction of a comprehensive evaluation and optimisation system.

A theoretical study of mathematical models of individualised learning has demonstrated the effectiveness of optimising partial trajectories through a system of recursive updating of model parameters. The introduction of the learning material complexity function *C (s, a)* into the modified Bellman equation formalised the process of adapting learning content to individual student characteristics. This approach, complemented by the mechanisms of coalition formation and collaborative learning through the theoretical and game components, creates a theoretical basis for increasing the effectiveness of individualised learning. In the context of adaptive learning technologies, the study by H.A. Alamri *et al.* (2021) is noteworthy. Their analysis of personalisation in a blended learning environment in higher education reflects the importance of adaptive mechanisms, which resonates with the theoretical model under consideration, especially in terms of optimising partial learning paths. However, the approach presented in the current study offers a deeper mathematical formalisation of adaptation processes through the integration of Markov processes and Bayesian optimisation, which allows for more accurate modelling of the dynamics of the learning process and individual characteristics of material perception.

M.L. Bernacki *et al*. (2021) raise fundamental questions about the goals and mechanisms of personalisation in their systematic review of research on personalised learning. Their conclusions regarding the need to explicitly define the goals of personalisation resonate with the multi-level model of strategic engagement considered in the current study. However, the theoretical model presented in this paper offers specific mathematical mechanisms for achieving these goals through formalising decision-making processes and optimising learning trajectories. An important contribution to interpreting the application of machine learning to precision education was made by H. Luan & C.-C. Tsai (2021). Their analysis confirmed the theoretical results of this study on the effectiveness of using mathematical models to individualise learning. Particularly significant is their conclusion about the need to fine-tune educational parameters, which in the current study is implemented through self-adaptation mechanisms and recursive updating of model parameters.

The analysis of the theoretical results and their comparison with current research in the field of mathematical modelling of individualised learning reveals both significant advantages of the proposed approach and potential directions for further development. The combination of Bayesian optimisation, dynamic programming, Markov processes and game theory creates a powerful theoretical foundation for modelling various aspects of the learning process. At the same time, it is worth noting that further development of the theoretical foundations of mathematical modelling of individualised learning requires experimental verification of the proposed models and mechanisms. Particular attention should be devoted to the study of the practical implementation of the considered mathematical models, their computational complexity and efficiency in different educational contexts. This will not only confirm the theoretical results but also highlight areas for further improvement of mathematical models of individualised learning.

Conclusions

The theoretical analysis of existing mathematical models of individualised learning has revealed a significant potential for integrating four key mathematical paradigms: Bayesian optimisation, dynamic programming, Markov processes and game theory. A comprehensive review of these approaches has shown a significant increase in the effectiveness of individualised learning, especially for students with different learning styles. In particular, the study of the partial trajectory optimisation method, which relies on recursive updating of model parameters through the analysis of intermediate results, demonstrated the possibility of achieving much more accurate parameter settings and ensuring smoother adaptation to the individual pace of learning by each student.

The theoretical study of the modified Bellman equation with the function of the complexity of the educational material revealed the fundamental principles of optimising long-term learning strategies, revealing a wider perspective on the dynamics of adapting the complexity of the content to the individual capabilities of each student. A comprehensive analysis of the stochastic nature of the learning process through an extended transition matrix revealed deep patterns in the dynamics of knowledge acquisition and was used to formalise the processes of forgetting and repeating material through differential equations with time-dependent coefficients. The study of collaborative learning mechanisms using the game-theoretic approach allowed not only to formalise the processes of forming learning coalitions and interaction between students but also to identify the synergistic effects of group learning through nonlinear functions of interaction between participants in the educational process. The proposed system of multidimensional evaluation of mathematical models covers a wide range of indicators – from basic knowledge acquisition to the development of higher-order metacognitive skills, which provides a comprehensive understanding of the effectiveness of the educational process and its components. Particular attention is paid to the analysis of long-term learning effects through a composite prediction function that incorporates the interaction of various aspects of the learning process and their impact on the formation of sustainable knowledge and skills.

The theoretical study has shown the high adaptability of the considered models to various educational contexts and their ability to consider a wide range of individual characteristics of students. The developed methodological foundations indicate the need to form an adaptive educational environment with a powerful decision-support system to determine optimal learning strategies. The analysis of the interaction of various components of the system and mechanisms of their coordination was emphasised, which identified the key factors of success of individualised learning and ways to improve its effectiveness.

The main limitation of this study was its theoretical nature, which indicates the urgent need for experimental verification of the models under consideration in a real educational environment with different groups of students and various learning contexts. Further research should address the development of effective methods for optimising the computational complexity of the integrated system, an in-depth study of the emotional and motivational aspects of the learning process, and the creation of reliable methods for validating the effectiveness of models in different educational and cultural contexts. Particular attention should be devoted to the study of the mechanisms of interaction between different mathematical models within a single decision-making system and the development of methods for their adaptation to specific educational tasks and goals.

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Conflict of Interest

The author declares no conflict of interest.

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Математичні моделі індивідуалізованого навчання, побудовані на теорії прийняття рішень

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Анотація. У дослідженні здійснено теоретичне обґрунтування та розроблення системи математичних моделей для індивідуалізації освітнього процесу на основі комплексної інтеграції методів теорії прийняття рішень. Розроблена система математичних моделей базується на метамоделі, що поєднує чотири математичні парадигми через матрицю взаємодії, елементи якої визначаються функцією когнітивної сумісності, часової узгодженості та ефективності взаємодії. Впровадження методу оптимізації часткових траєкторій, що спирається на рекурсивне оновлення параметрів моделі через аналіз проміжних результатів, дало змогу досягти точнішого налаштування параметрів та забезпечити плавну адаптацію до індивідуального темпу засвоєння матеріалу. Розроблена модифікація рівняння Беллмана з функцією складності навчального матеріалу допомогла формалізувати процес оптимізації довгострокових навчальних стратегій через врахування індивідуальних когнітивних особливостей. Аналіз стохастичної природи навчального процесу через розширену матрицю переходів дозволив математично описати процеси забування й повторення матеріалу за допомогою системи диференційних рівнянь з часозалежними коефіцієнтами, що враховують інтенсивність навчання та індивідуальні особливості пам'яті. Дослідження механізмів колаборативного навчання за допомогою теоретико-ігрового підходу виявило синергетичні ефекти групового навчання через нелінійні функції взаємодії учасників освітнього процесу та дозволило розробити методи формування оптимальних навчальних груп з урахуванням індивідуальних цілей. Запропонована система багатовимірного оцінювання, що реалізується через композитну цільову функцію, охоплює широкий спектр показників від базового засвоєння знань до розвитку метакогнітивних навичок вищого порядку, включаючи когнітивні, метакогнітивні та мотиваційні компоненти, що забезпечує надійний інструментарій для оцінки стійкості навчальних траєкторій та визначення рівня адаптивності системи до індивідуальних особливостей учнів

Ключові слова: адаптивні освітні системи; байєсівська оптимізація; функція Беллмана; марковські процеси; теоретико-ігровий підхід; когнітивні траєкторії